

# Constructions of Bent Functions from Two Known Bent Functions

Jennifer Seberry  
and  
Xian-Mo Zhang

Department of Computer Science  
The University of Wollongong  
Wollongong  
NSW 2522, AUSTRALIA

## Abstract

A  $(1, -1)$ -matrix will be called a bent type matrix if each row and each column are bent sequences. A similar description can be found in Carlisle M. Adams and Stafford E. Tavares, Generating and counting binary sequences, *IEEE Trans. Inform. Theory*, vol. 36, no. 5, pp. 1170-1173, 1990, in which the authors use the properties of bent type matrices to construct a class of bent functions. In this paper we give a general method to construct bent type matrices and show that the bent sequence obtained from a bent type matrix is a generalized result of the Kronecker product of two known bent sequences.

Also using two known bent sequences of length  $2^{2k-2}$  we can construct  $2^k - 2$  bent sequences of length  $2^{2k}$ , more than in the ordinary construction, which gives construct 10 bent sequences of length  $2^{2k}$  from two known bent sequences of length length  $2^{2k-2}$ .

Let  $V_n$  be the vector space of  $n$  tuples of elements from  $GF(2)$ . Let  $\alpha, \beta \in V_n$ . Write  $\alpha = (a_1, \dots, a_n)$ ,  $\beta = (b_1, \dots, b_n)$ , where  $a_i, b_i \in GF(2)$ . Write  $\langle \alpha, \beta \rangle = \sum_{j=1}^n a_j b_j$  for the scalar product of  $\alpha$  and  $\beta$ .

**Definition 1** We call the function  $h(x) = a_1 x_1 + \dots + a_n x_n + c$ ,  $a_j, c \in GF(2)$ , an *affine function*, in particular,  $h(x)$  will be called a *linear function* if  $c = 0$ .

**Definition 2** Let  $f(x)$  be a function from  $V_n$  to  $GF(2)$  (simply, a function on  $V_n$ ). If

$$2^{-\frac{n}{2}} \sum_{x \in V_n} (-1)^{f(x) + \langle \beta, x \rangle} = \pm 1,$$

for every  $\beta \in V_n$ . We call  $f(x)$  a *bent function* on  $V_n$ .

From Definition 2, bent functions on  $V_n$  only exist for even  $n$ . Bent functions were first introduced and studied by Rothaus [13]. Further properties, constructions and equivalence bounds for bent functions can be found in [2], [5], [7], [12], [16]. Kumar, Scholtz and Welch [6] defined and studied the bent functions from  $Z_q^n$  to  $Z_q$ . Bent functions are useful for digital communications, coding theory and cryptography [3], [1], [4], [7], [8], [10], [9], [11], [12].

We say  $\alpha = (a_1, \dots, a_n) < \beta = (b_1, \dots, b_n)$  if there exists  $k$ ,  $1 \leq k \leq n$ , such that  $a_1 = b_1, \dots, a_{k-1} = b_{k-1}$  and  $a_k = 0, b_k = 1$ . Hence we can order all vectors in  $V_n$  by the relation  $<$

$$\alpha_0 < \alpha_1 < \dots < \alpha_{2^n-1},$$

where

$$\begin{aligned} \alpha_0 &= (0, \dots, 0), \\ \alpha_1 &= (0, \dots, 1), \\ &\vdots \\ \alpha_{2^{n-1}-1} &= (0, 1, \dots, 1), \\ \alpha_{2^{n-1}} &= (1, 0, \dots, 0), \\ &\vdots \\ \alpha_{2^n-1} &= (1, 1, \dots, 1). \end{aligned}$$

**Definition 3** Let  $f(x)$  be a function from  $V_n$  to  $GF(2)$ . We call  $(-1)^{f(\alpha_0)}, (-1)^{f(\alpha_1)}, \dots, (-1)^{f(\alpha_{2^n-1})}$  the *sequence* of  $f(x)$ . We call the sequence of  $f(x)$  a *bent sequence* if  $f(x)$  is bent. A  $(1, -1)$ -sequence will be called an *affine sequence* a *(linear sequence)* if it is the sequence of an affine function (a linear function).

**Definition 4** A  $(1, -1)$ -matrix  $H$  of order  $h$  will be called an *Hadamard matrix* if  $HH^T = hI_h$ .

If  $h$  is the order of an Hadamard matrix then  $h$  is 1, 2 or divisible by 4 [15]. A special kind of Hadamard matrices defined as following will be relevant

**Definition 5** The *Sylvester-Hadamard matrix* (or *Walsh-Hadamard matrix*) of order  $2^n$ , denoted by  $H_n$ , is generated by the recursive relation

$$H_n = \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}, \quad n = 1, 2, \dots, \quad H_0 = 1.$$

Let  $f(x)$  be a function from  $V_n$  to  $GF(2)$ ,  $\xi$  be the sequence (regarded as a row vector) of  $f(x)$ . Then the following three conditions are equivalent

- (i)  $f(x)$  is bent,
- (ii)  $2^{-\frac{1}{2}n} H_n \xi^T$  is a  $(1, -1)$ -row vector,
- (iii) for any affine sequence  $l$   $\langle \xi, l \rangle = \pm 2^{\frac{1}{2}n}$ .

The equivalence of (i) and (ii) can be found in many references, for example, [2], [16]. Note that any affine sequence of length  $2^n$  is a row of  $\pm H_n$  (see subsection 2.3) thus (ii) and (iii) are equivalent.

**Definition 6** We call a  $(1, -1)$ -matrix of order  $2^m \times 2^n$  a *bent type matrix* if each row is a bent sequence of length of  $2^n$  and each column is a bent sequence of length of  $2^m$ .

For example,

$$\begin{bmatrix} + & + & + & - \\ + & + & - & + \\ - & - & - & + \\ + & + & - & + \end{bmatrix},$$

where  $+$  and  $-$  denote 1 and  $-1$  respectively, is a bent type matrix of order 4. A similar description can be found in [2, p. 1171].

**Definition 7** A  $(1, -1)$ -matrix of order  $2^m \times 2^n$  will be called *an affine type matrix* if each row is an affine sequence of length of  $2^n$  and each column is an affine sequence of length of  $2^m$ .

For example,

$$\begin{bmatrix} + & + & - & - & - & - & + & + \\ + & + & - & - & + & + & - & - \\ + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \end{bmatrix}$$

is an affine type matrix of order  $4 \times 8$ . Any Walsh-Hadamard matrix is an affine type matrix (see subsection 2.3).

**Definition 8** Let  $A_1$  and  $A_2$  be affine type matrices of order  $2^m \times 2^n$ . If  $A_2 = QA_1P$  where  $Q$  and  $P$  are diagonal matrices of order  $2^m$  and  $2^n$  whose diagonals consist of  $\pm 1$  we say  $A_1$  and  $A_2$  are *equivalent*.

For example  $\begin{bmatrix} - & + & + & - \\ - & + & + & - \\ + & - & - & + \\ + & - & - & + \end{bmatrix}$  and  $\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}$  are equivalent affine type matrices.

**Definition 9** We call each of the four  $(1, -1)$ -sequences of length 2  $++$ ,  $+-$ ,  $--$ ,  $-+$   $E^1$ -constructed. Recursively, suppose  $E^n$ -constructed has been defined for  $n = 1, \dots, k-1$ . The  $(1, -1)$ -sequence  $l$  will be said to be  $E^k$ -constructed if  $l = (l', \pm l')$  where  $l'$  is  $E^{k-1}$ -constructed.

## 1 Bent Type Matrices

### 1.1 Bent Type Matrices Constructed from Affine Type Matrices

**Lemma 1** Let  $b_0, b_1, \dots, b_{2^n-1}$  be a bent sequence and  $c_0, c_1, \dots, c_{2^n-1}$  be an affine sequence then  $b_0c_0, b_1c_1, \dots, b_{2^n-1}c_{2^n-1}$  is a bent sequence.

*Proof.* Let  $b_0, b_1, \dots, b_{2^n-1}$  be the sequence of a bent function  $f$  from  $V_n$  to  $GF(2)$  and  $c_0, c_1, \dots, c_{2^n-1}$  be the sequence of an affine function from  $V_n$  to  $GF(2)$ . Note that

$b_0c_0, b_1c_1, \dots, b_{2^n-1}c_{2^n-1}$  is the sequence of  $f + g$ . From Property 1 [6, p. 95]  $f + g$  is bent. This proves the lemma.  $\square$

Bent type matrices can be used to construct bent sequences. For convenience, we quote a part of the Theorem found in [2]

**Theorem 1** *Let  $B = (b_{ij})$  be a bent type matrix of order  $2^m \times 2^n$ . Write  $\beta_j = (b_{1j}, \dots, b_{2^m j})$ ,  $j = 1, \dots, 2^n$  and  $\alpha_i = (b_{i1} \dots b_{i2^n})$ ,  $i = 1, \dots, 2^m$ . Then both*

$$(2^{-\frac{1}{2}m} \beta_1 H_m, \dots, 2^{-\frac{1}{2}m} \beta_{2^n} H_m)$$

and

$$(2^{-\frac{1}{2}n} \alpha_1 H_n, \dots, 2^{-\frac{1}{2}n} \alpha_{2^m} H_n)$$

are bent sequences of length  $2^{m+n}$ .

*Proof.* The proof can be found in [2, p. 1171].  $\square$

Using the three equivalent conditions of bent functions in Section 1, both  $2^{-\frac{1}{2}m} \beta_j H_m$  and  $2^{-\frac{1}{2}n} \alpha_i H_n$  are bent sequences of length  $2^m$  and  $2^n$ . Hence Theorem 1 gives an example that the concatenation of some bent sequences is also bent. In general this is not true if some extra conditions are not satisfied. For example, each of  $+++-$ ,  $++-+$ ,  $+ - ++$ ,  $-+++$  is bent but the concatenation of the four sequences is not bent. The conditions for bent type matrices are restrictive. In this section we use affine type matrices to construct bent type matrices.

**Theorem 2** *Let  $A$  be an affine type matrix of order  $2^m \times 2^n$ ,  $P$  be a diagonal matrix of order  $2^n$  whose diagonal is a bent sequence of length  $2^n$ , say  $a_0, a_1, \dots, a_{2^n-1}$  and  $Q$  be a diagonal matrix of order  $2^m$  whose diagonal is a bent sequence of length  $2^m$ , say  $b_0, b_1, \dots, b_{2^m-1}$ . Then  $QAP$  is a bent type matrix of order  $2^m \times 2^n$ .*

*Proof.* Since each row of  $A$  is an affine sequence, by Lemma 1, each row of  $AP$  is a bent sequence. Note each column of  $AP$  is still an affine sequence. By Lemma 1, each column of  $QAP$  is a bent sequence. Note each row of  $QAP$  is still a bent sequence. This proves the theorem.  $\square$

To find the bent sequences using the special construction mentioned in Theorem 1, we first construct bent type matrices using Theorem 2. In particular, when the affine matrix  $A$  in Theorem 2 consists of only ones, the bent type matrix mentioned in Theorem 2 yields a bent sequence which is the Kronecker product (see [15]) of two bent sequences:  $2^{-\frac{1}{2}m} \beta_j H_m$  and  $2^{-\frac{1}{2}n} \alpha_i H_n$ . Thus we have reproved Theorem 1 [16] using a different method.

**Corollary 1** *Let  $\tau_n$  denote the number of different bent sequences on  $V_n$  with first entries  $+$  and  $\sigma_{m \times n}$  denote the number of inequivalent affine type matrices of order  $2^m \times 2^n$ . Then there exist at least  $\tau_m \tau_n \sigma_{m \times n}$  different bent type matrices of order  $2^m \times 2^n$ .*

*Proof.* We first note that for a fixed affine type matrix of order  $2^m \times 2^n$ , we can construct at least  $\tau_m \tau_n$  different bent type matrices of order  $2^m \times 2^n$  by using Theorem 2. Otherwise suppose  $B$  is an affine type matrix of order  $2^m \times 2^n$ ,  $Q_1 \neq Q_2$  or  $P_1 \neq P_2$  but  $Q_1 B P_1 = Q_2 B P_2$  where each  $Q_j$  and each  $P_j$  are the matrices mentioned in the proof of Theorem 2 whose first entries on the diagonals are  $+$ . Thus

$$Q_2 Q_1 B P_1 P_2 = B. \quad (1)$$

Note that both  $Q_2 Q_1$  and  $P_1 P_2$  are diagonal matrices whose diagonals consist of  $\pm 1$ . Let  $Q_2 Q_1 = \text{diag}(q_1, \dots, q_{2^k})$ ,  $P_1 P_2 = \text{diag}(p_1, \dots, p_{2^k})$ . Let  $B_1 = (b_1, \dots, b_{2^k})^T$  be the first column of  $B$ . Compare the first columns on each side of (1) then we have  $q_j b_j p_1 = b_j$ ,  $j = 1, \dots, 2^k$  thus  $q_j = p_1$ ,  $j = 1, \dots, 2^k$  and thus  $Q_2 Q_1 = \pm I_{2^k}$  according as  $p_1 = \pm 1$ . Hence  $Q_2 Q_1 = e I_{2^m}$  and  $P_1 P_2 = e I_{2^n}$  where  $e = \pm 1$ . Since the first entries on the diagonals of  $Q_1, Q_2, P_1, P_2$  are  $+$ ,  $Q_1 = Q_2$  and  $P_1 = P_2$ . This contradicts to the assumption that  $Q_1 \neq Q_2$  or  $P_1 \neq P_2$ .

Secondly we note that if  $B_1$  and  $B_2$  are inequivalent affine type matrices of order  $2^m \times 2^n$ , there exist no  $Q_1, Q_2, P_1, P_2$  as mentioned in Theorem 2 such that  $Q_1 B_1 P_1 = Q_2 B_2 P_2$ . Otherwise we would have  $Q_2 Q_1 B_1 P_1 P_2 = B_2$ . This contradicts the assumption that  $B_1$  and  $B_2$  are inequivalent. Hence we have established the corollary.  $\square$

## 1.2 Constructing Affine Type Matrices

**Lemma 2** Write  $H_n = \begin{bmatrix} l_0 \\ l_1 \\ \vdots \\ l_{2^n-1} \end{bmatrix}$  where  $l_i$  is a row of  $H_n$ . Then  $l_i$  is the sequence of a linear function on  $V_n$ .

*Proof.* The proof can be found in [14].  $\square$

We can now established

**Theorem 3** An  $(1, -1)$ -matrix of order  $2^m \times 2^n$  is an affine type matrix if and only if each row is  $E^n$ -constructed and each column is  $E^m$ -constructed.

*Proof.* Note that  $H_n$  has  $2^n$  rows and there exist  $2^n$  linear sequences of length  $2^n$ . By Lemma 2 each linear sequence is a row of  $H_n$  and thus each affine sequence is a row of  $\pm H_n$ . By the Definition of  $H_n$  each row of  $H_n$  and is  $E^n$ -constructed. Hence each affine sequence is  $E^n$ -constructed. On the other hand, there exist  $2^{n+1}$   $E^n$ -constructed  $(1 -1)$ -sequences and  $2^{n+1}$  affine sequences. Thus each  $E^n$ -constructed  $(1 -1)$ -sequences is affine.  $\square$

**Theorem 4** Let  $A_1$  be an affine type matrix of order  $2^{m_1} \times 2^{n_1}$  with rank  $r_1$  and  $A_2$  be an affine type matrix of order  $2^{m_2} \times 2^{n_2}$  with rank  $r_2$ . Then  $A_1 \times A_2$  is an affine type matrix of order  $2^{m_1+m_2} \times 2^{n_1+n_2}$  with rank  $r_1 r_2$ , where  $\times$  is the Kronecker product.

*Proof.* Note that each row of  $A_1 \times A_2$  is  $E^{n_1+n_2}$ -constructed and each column of  $A_1 \times A_2$  is  $E^{m_1+m_2}$ -constructed. Hence by Theorem 3,  $A_1 \times A_2$  is an affine type matrix.

Let  $C_1$  be the invertible submatrix of order  $r_1$  and  $C_2$  be the invertible submatrix of order  $r_2$ . Hence by (25) of [16, p. 114],  $C_1 \times C_2$  is invertible and thus the rank of  $A_1 \times A_2$  is at least  $r_1 r_2$ .

On the other hand, since the ranks of  $A_1$  and  $A_2$  are  $r_1$  and  $r_2$  respectively, write suppose  $\alpha_1, \dots, \alpha_{r_1}$  for the linearly independent row vectors of  $A_1$ , and  $\beta_1, \dots, \beta_{r_2}$  for the linearly independent column vectors of  $A_2$ . Note that any row vector of  $A_1$  is a linear combination of  $\alpha_1, \dots, \alpha_{r_1}$  and any row vector of  $A_2$  is a linear combination of  $\beta_1, \dots, \beta_{r_2}$ . Any row vector of  $A_1 \times A_2$  can be written as  $\alpha \times \beta$ , where  $\alpha$  is a row vector of  $A_1$  and  $\beta$  is a row vector of  $A_2$ . Write  $\alpha = \sum_{j=1}^{r_1} a_j \alpha_j$  and  $\beta = \sum_{j=1}^{r_2} b_j \beta_j$ , where each  $a_j$  and  $b_j \in GF(2)$ . Hence

$$\alpha \times \beta = \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} a_i b_j (\alpha_i \times \beta_j).$$

This proves that the rank of  $A_1 \times A_2$  is at most  $r_1 r_2$  and hence it is exactly  $r_1 r_2$ .  $\square$

**Corollary 2** (i) let  $A$  be an affine type matrix of order  $2^m \times 2^n$  with rank  $r$  and  $\alpha$  be the row vector of an affine sequence of length  $2^s$  then both  $\alpha \times A$  and  $A \times \alpha$  are affine type matrix of order  $2^m \times 2^{n+s}$  with rank  $r$ ,

(ii) let  $\alpha$  be the row vector of an affine sequence of length  $2^s$  then both  $\alpha \times H_n$  and  $H_n \times \alpha$  are affine type matrices of order  $2^n \times 2^{n+s}$  with rank  $2^n$ , where  $H_n$  is a Walsh-Hadamard matrix,

(iii) let  $\alpha$  be the row vector of an affine sequence of length  $2^s$  and  $\beta$  be the row vector of an affine sequence of length  $2^t$  then  $\alpha \times \beta^T$  is an affine type matrix of order  $2^t \times 2^s$  with rank 1.

**Theorem 5** For any integers  $k, n, m, 0 \leq k \leq n \leq m$ , there exists at least  $(2^k - 1)!$  inequivalent (under the meaning in Definition 8) affine type matrices of order  $2^m \times 2^n$  with rank  $2^k$ .

*Proof.* Write Walsh-Hadamard matrix  $H_k = [h_1 \cdots h_{2^k}]$  where each  $h_j$  is the column vector of  $H_k$ . We first prove that any two  $[h_1 h_{j_2} \cdots h_{j_{2^k}}]$  and  $[h_1 h_{i_2} \cdots h_{i_{2^k}}]$  are inequivalent if  $j_2, \dots, j_{2^k}$  and  $i_2, \dots, i_{2^k}$  are two different rearrangements of  $2, \dots, 2^k$ . Otherwise if there exist diagonal matrices as mentioned in Definition 8, say  $Q = \text{diag}(q_1, \dots, q_{2^k})$ ,  $P = \text{diag}(p_1, \dots, p_{2^k})$ , then  $Q = \pm I_{2^k}$ ,  $P = \pm I_{2^k}$ , since

$$Q[h_1 h_{j_2} \cdots h_{j_{2^k}}]P = [h_1 h_{i_2} \cdots h_{i_{2^k}}], \quad (2)$$

and comparing the first columns on each side of (2), we have  $q_j a_j p_1 = a_j$  where  $(a_1, \dots, a_{2^k})^T = h_1$ , thus  $q_j = p_1$ ,  $j = 1, \dots, 2^k$  and thus  $Q = \pm I_{2^k}$  according as  $p_1 = \pm 1$ . By the same reasoning we can prove that  $P = \pm I_{2^k}$ , according as  $q_1 = \pm 1$ . On the other hand, there exists an integer  $t, 2 \leq t \leq 2^k$  such as  $j_t \neq i_t$  and thus  $h_{j_t} \neq h_{i_t}$ . We note that (2) cannot hold by comparing  $h_{j_t}$  and  $h_{i_t}$ . This proves the above statement.

Let  $R$  be the matrix of order  $2^{m-k} \times 2^{n-k}$  with elements ones. By Theorem 4

$[h_1 h_{j_2} \cdots h_{j_{2^k}}] \times R$  is an affine type matrix of order  $2^m \times 2^n$  with rank  $2^k$ . Permuting  $j_2, \dots, j_{2^k}$  we obtain  $(2^k - 1)!$  inequivalent matrices of this kind.  $\square$

Note that  $0! = 1$  in Theorem 5.

**Corollary 3** *For any positive integers  $n$  and  $m$ ,  $n \leq m$ , there exist at least  $\sum_{k=0}^n (2^k - 1)!$  inequivalent (within the meaning of Definition 8) affine type matrix of order  $2^m \times 2^n$ .*

*Proof.* We note that if two matrices have different ranks they are inequivalent within the meaning of Definition 8.  $\square$

**Corollary 4** *For any positive integers  $n \leq m$  there exists at least  $\tau_n \tau_m \sum_{k=0}^n (2^k - 1)!$  different bent type matrices of order  $2^m \times 2^n$ .*

*Proof.* By Corollary 3  $\sigma_{m \times n} \geq \sum_{k=1}^n (2^k - 1)!$ . Using Corollary 1 we have proved the corollary.  $\square$

## 2 Combination of Two Known Bent Sequences

### 2.1 Enumeration of Nondegenerate Linear Transformations

We replace the real numbers  $1, 2, \dots, 2^n$  by the vectors

$$\alpha_0 = (0, \dots, 0), \alpha_1 = (0, \dots, 0, 1), \dots, \alpha_{2^n-1} = (1, 1, \dots, 1) \in V_n$$

respectively. Let  $\varphi$  be nondegenerate linear transformation on  $V_n$ . Set  $\beta_j = \varphi(\alpha_j)$ ,  $j = 0, 1, \dots, 2^n - 1$ .

**Lemma 3** *If  $e_1, e_2, \dots, e_{2^n}$  i.e.  $e_{\alpha_0}, e_{\alpha_1}, \dots, e_{\alpha_{2^n-1}}$  is an affine sequence then  $e_{\beta_0}, e_{\beta_1}, \dots, e_{\beta_{2^n-1}}$  is also an affine sequence.*

*Proof.* Let  $e_{\alpha_0}, e_{\alpha_1}, \dots, e_{\alpha_{2^n-1}}$  be the sequence of the affine function  $h(x_1, \dots, x_n)$  on  $V_n$ . Set  $h(\varphi(x_1, \dots, x_n)) = g(x_1, \dots, x_n)$  thus  $h(\varphi(\alpha_j)) = g(\alpha_j)$  i.e.  $h(\beta_j) = g(\alpha_j)$  and thus  $e_{\beta_j} = (-1)^{h(\beta_j)} = (-1)^{g(\alpha_j)}$ . Since  $g(x_1, \dots, x_n)$  is an affine function the sequence of  $g$  i.e.  $e_{\beta_0}, e_{\beta_1}, \dots, e_{\beta_{2^n-1}}$  is an affine sequence.  $\square$

**Lemma 4** *There exist exactly  $\prod_{j=0}^{n-1} (2^n - 2^j)$  nondegenerate linear transformations on  $V_n$ .*

*Proof.* An equivalent statement is that there exist exactly  $\prod_{j=0}^{n-1} (2^n - 2^j)$  non-degenerate

matrices of order  $n$  over  $GF(2)$ . Write  $D = \begin{bmatrix} D_1 \\ \vdots \\ D_{2^n} \end{bmatrix}$ , a non-degenerate matrix of order

$n$  over  $\text{GF}(2)$ , where  $D_i$  is the  $i$ -th row of  $D$ . Note that  $D_1$  has  $2^n - 1$  choices (excluding the case that  $D_1$  is the zero vector). After  $D_1$  is fixed  $D_2$  has  $2^n - 2$  choices (excluding  $D_2 = d_1 D_1$  where  $d_1 = 0, 1$ ). After  $D_1$  and  $D_2$  are fixed  $D_3$  has  $2^n - 2^2$  choices (excluding  $D_3 = d_1 D_1 + d_2 D_2$ , where  $d_1, d_2 = 0, 1$ ). Continuing this reasoning, after  $D_1, \dots, D_{n-1}$  have been fixed  $D_n$  has  $2^n - 2^{n-1}$  choices (excluding  $D_n = \sum_{j=1}^{n-1} d_j D_j$ , where each  $d_j = 0, 1$ ). In total  $D$  has  $\prod_{j=0}^{n-1} (2^n - 2^j)$  different choices.  $\square$

**Lemma 5** (i) *All nondegenerate linear transformations on  $V_n$  can be divided into  $2^n - 1$  disjoint classes  $\Omega_1, \dots, \Omega_{2^n-1}$  such that  $\varphi_1$  and  $\varphi_2$  are in the same class if and only if  $\{\varphi_1(\alpha_0), \dots, \varphi_1(\alpha_{2^n-1-1})\} = \{\varphi_2(\alpha_0), \dots, \varphi_2(\alpha_{2^n-1-1})\}$ ,*

(ii)  $|\Omega_j| = 2^{n-1} \prod_{j=0}^{n-2} (2^{n-1} - 2^j)$ ,  $j = 1, \dots, 2^n - 1$ .

*Proof.* Fix a nondegenerate linear transformation on  $V_n$ , say  $\varphi_0$ . Write  $\varphi_0(\alpha_j) = \beta_j^0$ ,  $j = 1, \dots, 2^n - 1$ .

We now count  $\varphi$  such that  $\varphi$  and  $\varphi_0$  are in the same class i.e.  $\{\varphi(\alpha_0), \dots, \varphi(\alpha_{2^n-1-1})\} = \{\varphi_0(\alpha_0), \dots, \varphi_0(\alpha_{2^n-1-1})\} = \{\beta_0, \dots, \beta_{2^n-1-1}\}$ . This counting is equivalent to counting the nondegenerate linear transformations on  $V_n$ , say  $\psi$ , such that  $\{\psi(\beta_0), \dots, \psi(\beta_{2^n-1-1})\} = \{\beta_0, \dots, \beta_{2^n-1-1}\}$  because if we set  $\varphi = \psi\varphi_0$  then  $\{\varphi(\alpha_0), \dots, \varphi_1(\alpha_{2^n-1-1})\} = \{\psi\varphi_0(\alpha_0), \dots, \psi\varphi_0(\alpha_{2^n-1-1})\} = \{\psi(\beta_0), \dots, \psi(\beta_{2^n-1-1})\} = \{\beta_0, \dots, \beta_{2^n-1-1}\} = \{\varphi_0(\alpha_0), \dots, \varphi_0(\alpha_{2^n-1-1})\}$ . Since  $\{\alpha_0, \dots, \alpha_{2^n-1-1}\}$  contains  $\alpha_1, \alpha_2, \alpha_{2^2}, \dots, \alpha_{2^{n-2}}$  but contains no  $\alpha_j$ ,  $j = 2^{n-1}, \dots, \alpha_{2^n-1}$ , the rank of  $\{\alpha_0, \dots, \alpha_{2^n-1-1}\}$  is  $n - 1$ . Note that any nondegenerate linear transformation preserves the rank of any set of vectors thus the rank of  $\{\beta_0, \dots, \beta_{2^n-1-1}\}$  is also  $n - 1$ . Suppose  $\beta_{j_1}, \dots, \beta_{j_{n-1}} \in \{\beta_0, \dots, \beta_{2^n-1-1}\}$  is a basis for  $\{\beta_0, \dots, \beta_{2^n-1-1}\}$ . Add an appropriate vector in  $V_n$ , say  $\gamma$ , such that  $\beta_{j_1}, \dots, \beta_{j_{n-1}}, \gamma$  form a basis of  $V_n$ .

We now determine  $\psi$  such that  $\{\psi(\beta_0), \dots, \psi(\beta_{2^n-1-1})\} = \{\beta_0, \dots, \beta_{2^n-1-1}\}$ . For this purpose a necessary and sufficient condition is

$$\begin{aligned} \psi(\beta_{j_1}) &= c_{11}\beta_{j_1} + c_{12}\beta_{j_2} + \dots + c_{1n-1}\beta_{j_{n-1}} \\ \psi(\beta_{j_2}) &= c_{21}\beta_{j_1} + c_{22}\beta_{j_2} + \dots + c_{2n-1}\beta_{j_{n-1}} \\ &\vdots \\ \psi(\beta_{j_{n-1}}) &= c_{n-11}\beta_{j_1} + c_{n-12}\beta_{j_2} + \dots + c_{n-1n-1}\beta_{j_{n-1}} \\ \psi(\gamma) &= d_1\beta_{j_1} + d_2\beta_{j_2} + \dots + d_{n-1}\beta_{j_{n-1}} + e\gamma \end{aligned}$$

where  $(c_{ij})$  is a nondegenerate matrix of order  $n - 1$  on  $V_{n-1}$  and  $e = 1$  since  $\psi$  is a nondegenerate linear transformation. By Lemma 4  $(c_{ij})$  has  $\prod_{j=0}^{n-2} (2^{n-1} - 2^j)$  choices. On the other hand  $(d_1, \dots, d_{n-1})$  has  $2^{n-1}$  choices. In total  $\psi$  has  $2^{n-1} \prod_{j=0}^{n-2} (2^{n-1} - 2^j)$  choices. This proves that  $|\Omega_j| = 2^{n-1} \prod_{j=0}^{n-2} (2^{n-1} - 2^j)$ ,  $j = 1, \dots, 2^n - 1$ . By Lemma 4 there exists  $\prod_{j=0}^{n-1} (2^n - 2^j)$  nondegenerate linear transformations on  $V_n$ . Thus we have  $\prod_{j=0}^{n-1} (2^n - 2^j) / 2^{n-1} \prod_{j=0}^{n-2} (2^{n-1} - 2^j) = 2^n - 1$  disjoint classes.  $\square$

## 2.2 Combination of Two Known Bent Functions

In this section we replace  $1, 2, \dots, 2^{2k-1}$  by vectors in  $V_{2k-1}$ :  $\alpha_0 = (0, \dots, 0)$ ,  $\alpha_1 = (0, \dots, 0, 1)$ ,  $\dots$ ,  $\alpha_{2^{2k-1}-1} = (1, 1, \dots, 1)$  respectively.

Let  $\varphi$  be nondegenerate linear transformation on  $V_{2^{2k-1}}$ . Set  $\beta_j = \varphi(\alpha_j)$ ,  $j = 0, 1, \dots, 2^{2k-1} - 1$ . Suppose  $\xi_1 = (a_1, \dots, a_{2^{2k-2}})$  and  $\xi_2 = (b_1, \dots, b_{2^{2k-2}})$  are two bent sequences of length  $2^{2k-2}$ . We now construct a  $(1 \ -1)$ -sequence of length  $2^{2k}$ , denoted by  $\eta = (\eta_1, \eta_2)$  where each  $\eta_j$  is of length  $2^{2k-1}$ , by using  $\xi_1, \xi_2$  and  $\varphi$ .

**Construction 1** Let the  $\beta_0$ -th, the  $\beta_1$ -th,  $\dots$ , and the  $\beta_{2^{2k-2}-1}$ -th entries of  $\eta_1$  be  $a_1, a_2, \dots, a_{2^{2k-1}}$  respectively and let the  $\beta_{2^{2k-2}}$ -th, the  $\beta_{2^{2k-2}+1}$ -th,  $\dots$ , and the  $\beta_{2^{2k-1}-1}$ -th entries of  $\eta_1$  be  $b_1, b_2, \dots, b_{2^{2k-1}}$  respectively. Next let the  $\beta_0$ -th, the  $\beta_1$ -th,  $\dots$ , and the  $\beta_{2^{2k-2}-1}$ -th entries of  $\eta_2$  be  $a_1, a_2, \dots, a_{2^{2k-1}}$  respectively and let the  $\beta_{2^{2k-2}}$ -th, the  $\beta_{2^{2k-2}+1}$ -th,  $\dots$ , and the  $\beta_{2^{2k-1}-1}$ -th entries of  $\eta_2$  be  $-b_1, -b_2, \dots, -b_{2^{2k-1}}$  respectively. Set  $\eta = (\eta_1, \eta_2)$ .

**Lemma 6**  $\eta$ , in Construction 1, is a bent sequence of length  $2^{2k}$ .

*Proof.* Let  $L$  be an affine sequence of length  $2^{2k}$ . By Theorem 3  $L = (l, \pm l)$  where  $l$  is an affine sequence of length  $2^{2k-1}$ . Write  $l = (e_1, e_2, \dots, e_{2^{2k-1}})$  i.e.  $l = (e_{\alpha_0}, e_{\alpha_1}, \dots, e_{\alpha_{2^{2k-1}-1}})$ . Write  $l = (l_1, l_2)$  where each  $l_j$  is of length  $2^{2k-2}$ . By Theorem 3 each  $l_j$  is an affine sequence of length  $2^{2k-2}$  and  $l_2 = \pm l_1$ .

We now consider  $\langle \eta, L \rangle = \langle \eta_1, l_1 \rangle + \langle \eta_2, l_2 \rangle$ .

Case 1:  $L = (l, l)$ . By Construction 1

$$\langle \eta, L \rangle = \langle \eta_1, l \rangle + \langle \eta_2, l \rangle$$

where

$$\langle \eta_1, l \rangle = \sum_{j=1}^{2^{2k-2}} a_j e_{\beta_{j-1}} + \sum_{j=1}^{2^{2k-2}} b_j e_{\beta_{2^{2k-2}+j-1}}$$

and

$$\langle \eta_2, l \rangle = \sum_{j=1}^{2^{2k-2}} a_j e_{\beta_{j-1}} - \sum_{j=1}^{2^{2k-2}} b_j e_{\beta_{2^{2k-2}+j-1}}.$$

Thus

$$\langle \eta, L \rangle = 2 \sum_{j=1}^{2^{2k-2}} a_j e_{\beta_{j-1}}. \quad (3)$$

Write  $l^* = (e_{\beta_0}, e_{\beta_1}, \dots, e_{\beta_{2^{2k-1}-1}})$ , by Lemma 3, it is an affine sequence of length  $2^{2k-1}$ . Write  $l^* = (l_1^*, l_2^*)$  where each  $l_j^*$  is of length  $2^{2k-2}$ . By Theorem 3 each  $l_j^*$  is an affine sequence of length  $2^{2k-2}$ .

Thus (3) becomes  $\langle \eta, L \rangle = 2 \langle \xi_1, l_1^* \rangle$ . Note that  $\xi_1$  is a bent sequence of length  $2^{2k-2}$  and  $l_1^*$  is an affine sequence of length  $2^{2k-2}$ . Thus  $\langle \xi_1, l_1^* \rangle = \pm 2^{k-1}$  and hence  $\langle \eta, L \rangle = \pm 2^k$ .

Case 2:  $L = (l, -l)$ . By Construction 1

$$\langle \eta, L \rangle = \langle \eta_1, l \rangle - \langle \eta_2, l \rangle$$

where

$$\langle \eta_1, l \rangle = \sum_{j=1}^{2^{2k-2}} a_j e_{\beta_{j-1}} + \sum_{j=1}^{2^{2k-2}} b_j e_{\beta_{2^{2k-2}+j-1}}$$

and

$$\langle \eta_2, l \rangle = \sum_{j=1}^{2^{2k-2}} a_j e_{\beta_{j-1}} - \sum_{j=1}^{2^{2k-2}} b_j e_{\beta_{2^{2k-2}+j-1}}.$$

Thus

$$\langle \eta, L \rangle = 2 \sum_{j=1}^{2^{2k-2}} b_j e_{\beta_{2^{2k-2}+j-1}} = 2 \langle \xi_2, l_2^* \rangle. \quad (4)$$

Note that  $\xi_2$  is a bent sequence of length  $2^{2k-2}$  and  $l_2^*$  is an affine sequence of length  $2^{2k-2}$ . Thus  $\langle \xi_2, l_2^* \rangle = \pm 2^{k-1}$  and hence (4) becomes  $\langle \eta, L \rangle = \pm 2^k$ .

Since  $L$  is arbitrary, by the three equivalent conditions of bent functions,  $\eta$  is a bent sequence.  $\square$

**Construction 2** let the  $\beta_0$ -th, the  $\beta_1$ -th,  $\dots$ , and the  $\beta_{2^{2k-2}-1}$ -th entries of  $\eta_1$  be  $a_1, a_2, \dots, a_{2^{2k-1}}$  respectively and let the  $\beta_{2^{2k-2}}$ -th, the  $\beta_{2^{2k-2}+1}$ -th,  $\dots$ , and the  $\beta_{2^{2k-1}-1}$ -th entries of  $\eta_1$  be  $b_1, b_2, \dots, b_{2^{2k-1}}$  respectively.

Next let the  $\beta_0$ -th, the  $\beta_1$ -th,  $\dots$ , and the  $\beta_{2^{2k-2}-1}$ -th entries of  $\eta_2$  be  $-a_1, -a_2, \dots, -a_{2^{2k-1}}$  respectively and let the  $\beta_{2^{2k-2}}$ -th, the  $\beta_{2^{2k-2}+1}$ -th,  $\dots$ , and the  $\beta_{2^{2k-1}-1}$ -th entries of  $\eta_2$  be  $b_1, b_2, \dots, b_{2^{2k-1}}$  respectively.

Set  $\eta = (\eta_1 \ \eta_2)$ .

**Lemma 7**  $\eta$ , in Construction 2, is a bent sequence of length  $2^{2k}$ .

*Proof.* The proof is similar to the proof of Lemma 6.  $\square$

### 2.3 Enumeration of Bent Sequences by Construction 1 and 2

**Lemma 8** Let  $\Xi_{2^k}^1$  denote the set of bent sequences of length  $2^{2k}$  obtained via Construction 1 and  $\Xi_{2^k}^2$  denote the set of bent sequences of length  $2^{2k}$  obtained via Construction 2. Then  $\Xi_{2^k}^1 \cap \Xi_{2^k}^2 = \phi$  where  $\phi$  denotes the empty set.

*Proof.* Suppose we construct the bent sequence of length  $2^{2k}$ , say  $\eta = (\eta_1, \eta_2)$ , by using the bent sequences  $\xi_1 = (a_1, \dots, a_{2^{2k-2}})$ ,  $\xi_2 = (b_1, \dots, b_{2^{2k-2}})$  and the nondegenerate linear transformation on  $V_{2^{k-1}}$ , denoted by  $\varphi$ , in Construction 1. Similarly we suppose in Construction 2 we construct a bent sequence of length  $2^{2k}$ , say  $\eta' = (\eta'_1, \eta'_2)$ , by using bent sequences  $\xi_1 = (a'_1, \dots, a'_{2^{2k-2}})$ ,  $\xi_2 = (b'_1, \dots, b'_{2^{2k-2}})$  and a nondegenerate linear transformation on  $V_{2^{k-1}}$ , denoted by  $\varphi'$ .

Set  $\beta_j = \varphi(\alpha_j)$ ,  $\beta'_j = \varphi'(\alpha_j)$  where  $j = 0, 1, \dots, 2^{2k-1} - 1$ . Note that  $\beta_0 = \varphi(\alpha_0)$ ,  $\beta'_0 = \varphi'(\alpha_0)$  and  $\alpha_0 = (0, 0, \dots, 0)$  thus  $\beta_0 = \beta'_0 = (0, 0, \dots, 0)$  since both  $\varphi$  and  $\varphi'$  are linear transformations.

In Construction 1  $a_1$  occurs in the  $\beta_0$ -th place of  $\eta_1$  also  $a_1$  occurs in the  $\beta_0$ -th place of  $\eta_2$ . Thus the first entries in  $\eta_1$  and  $\eta_2$  are the same.

In Construction 2  $a'_1$  occurs in the  $\beta_0$ -th place of  $\eta'_1$  also  $-a'_1$  occurs in the  $\beta_0$ -th place of  $\eta'_2$ . Thus the first entries in  $\eta'_1$  and  $\eta'_2$  are negatives each other. This proves that  $\eta \neq \eta'$ . Since both  $\eta$  and  $\eta'$  are arbitrary  $\Xi_{2k}^1 \cap \Xi_{2k}^2 = \phi$ .  $\square$

By Lemma 5 we divide all nondegenerate linear transformations on  $V_{2k-1}$  into  $2^{2k-1} - 1$  disjoint classes:  $\Omega_1, \dots, \Omega_{2^{2k-1}-1}$  such that  $\varphi_1$  and  $\varphi_2$  are in the same class if and only if  $\{\varphi_1(\alpha_0), \dots, \varphi_1(\alpha_{2^{2k-2}-1})\} = \{\varphi_2(\alpha_0), \dots, \varphi_2(\alpha_{2^{2k-2}-1})\}$ . We fix a  $\varphi_s \in \Omega_s$ ,  $s = 1, \dots, 2^{2k-1} - 1$ .

**Lemma 9** *Suppose we construct the bent sequence of length  $2^{2k}$ , say  $\eta = (\eta_1, \eta_2)$ , by using the bent sequences  $\xi_1 = (a_1, \dots, a_{2^{2k-2}})$ ,  $\xi_2 = (b_1, \dots, b_{2^{2k-2}})$  and the nondegenerate linear transformation on  $V_{2k-1}$ , denoted by  $\varphi_s$  where  $\varphi_s \in \Omega_s$ , in Construction 1 (2). Also in Construction 1 (2) we construct a bent sequence of length  $2^{2k}$ , say  $\eta' = (\eta'_1, \eta'_2)$ , by using bent sequences  $\xi_1 = (a'_1, \dots, a'_{2^{2k-2}})$ ,  $\xi_2 = (b'_1, \dots, b'_{2^{2k-2}})$  and a nondegenerate linear transformation on  $V_{2k-1}$ , denoted by  $\varphi_t$  where  $\varphi_t \in \Omega_t$ . If  $t \neq s$  then  $\eta \neq \eta'$ .*

*Proof.* Set  $\beta_j = \varphi_s(\alpha_j)$ ,  $\beta'_j = \varphi_t(\alpha_j)$  where  $j = 0, 1, \dots, 2^{2k-1} - 1$ . Since  $\{\varphi_s(\alpha_0), \dots, \varphi_s(\alpha_{2^{2k-2}-1})\} \neq \{\varphi_t(\alpha_0), \dots, \varphi_t(\alpha_{2^{2k-2}-1})\}$  i.e.  $\{\beta_0, \dots, \beta_{2^{2k-2}-1}\} \neq \{\beta'_0, \dots, \beta'_{2^{2k-2}-1}\}$  there exists a  $\beta$  such that  $\beta \in \{\beta_0, \dots, \beta_{2^{2k-2}-1}\}$  but  $\beta \notin \{\beta'_0, \dots, \beta'_{2^{2k-2}-1}\}$ .

In Construction 1 we note that  $\beta \in \{\beta_0, \dots, \beta_{2^{2k-2}-1}\}$  and we can suppose  $a_{i_0}$  occurs in the  $\beta$ -th place of  $\eta_1$  and  $a_{i_0}$  also occurs in the  $\beta$ -th place of  $\eta_2$  thus the entry in the  $\beta$ -th place of  $\eta_1$  and the entry in the  $\beta$ -th place of  $\eta_2$  are the same.

For  $\eta'$ , in Construction 1, we note that  $\beta \notin \{\beta'_0, \dots, \beta'_{2^{2k-2}-1}\}$  thus  $\beta \in \{\beta'_{2^{2k-2}}, \dots, \beta'_{2^{2k-1}-1}\}$  and we can suppose  $b_{j_0}$  occurs in the  $\beta$ -th place of  $\eta'_1$  and  $-b'_{j_0}$  occurs in the  $\beta$ -th place of  $\eta'_2$  thus the entry in the  $\beta$ -th place of  $\eta'_1$  and the entry in the  $\beta$ -th place of  $\eta'_2$  are negatives of each other. This proves  $\eta \neq \eta'$ . Similarly we can prove the lemma for Construction 2.  $\square$

**Lemma 10** *We fix a  $\varphi_s \in \Omega_s$ . Suppose we construct the bent sequence of length  $2^{2k}$ , say  $\eta = (\eta_1, \eta_2)$ , by using the bent sequences  $\xi_1 = (a_1, \dots, a_{2^{2k-2}})$ ,  $\xi_2 = (b_1, \dots, b_{2^{2k-2}})$  and the nondegenerate linear transformation on  $V_{2k-1}$ , say  $\varphi_s$ , in Construction 1 (2). Also in Construction 1 (2) we construct a bent sequence of length  $2^{2k}$ , say  $\eta' = (\eta'_1, \eta'_2)$ , by using bent sequences  $\xi_1 = (a'_1, \dots, a'_{2^{2k-2}})$ ,  $\xi_2 = (b'_1, \dots, b'_{2^{2k-2}})$  and the same nondegenerate linear transformation  $\varphi_s$ . If  $(\xi'_1, \xi'_2) \neq (\xi_1, \xi_2)$  then  $\eta \neq \eta'$ .*

*Proof.* Without any loss of generality suppose  $a_{j_0} \neq a'_{j_0}$  for some  $j_0$ . By Construction 1  $a_{j_0}$  occurs in the  $\beta_{j_0-1}$ -th place of  $\eta_1$ .

On the other hand, by Construction 1,  $a'_{j_0}$  occurs in the  $\beta_{j_0-1}$ -th place of  $\eta'_1$ . Thus  $\eta_1 \neq \eta'_1$  and thus  $\eta \neq \eta'$ . Similarly we can prove the lemma for Construction 2.  $\square$

**Theorem 6** (i) Using two bent sequences of length  $2^{2k-2}$ , say  $\xi_1$  and  $\xi_2$ , we can construct  $2^{2k} - 2$  different bent sequences of length  $2^{2k}$ ,

(ii) let  $\tau_{2k}$  denote the number of the bent sequences of length  $2^{2k}$  then  $\tau_{2k} \geq (2^{2k} - 2)\tau_{2k-2}^2$  for  $k \geq 2$ .

*Proof.* (i) For the two bent sequences of length of  $2^{2k-2}$  in Construction 1 (2),  $\varphi$  has  $2^{2k-1} - 1$  choices. By Lemma 9 we can construct  $2^{2k-1} - 1$  different bent sequences from the two known bent sequences of length of  $2^{2k-2}$ . By Lemma 8 we have  $2^{2k} - 2$  different bent sequences of length of  $2^{2k}$  in Construction 1 and 2 in total.

(ii) Two bent sequences of length  $2^{2k} - 2$  have  $\tau_{2k-2}^2$  choices. By Lemma 10 and (i) of the theorem  $\tau_{2k} \geq (2^{2k} - 2)\tau_{2k-2}^2$  for  $k \geq 2$ .  $\square$

We note that (i) of Theorem 6 gives many more bent sequences of length  $2^{2k}$  from two known bent sequences of length  $2^{2k-2}$  than the ordinary construction, which gives 10 bent sequences of length  $2^{2k}$  from two known bent sequences of length  $2^{2k-2}$  (see [2]).

## 2.4 Examples

**Example 1** Since  $\tau_2 = 8$ , by Theorem 1,  $\tau_4 \geq (2^4 - 2)8^2 = 896$  and  $\tau_6 \geq (2^6 - 2)\tau_4^2 = 62 \cdot 896^2 = 62 \cdot 802816 = 49774592$ .

Previously Adams and Tavares [2] estimated 48201728 as the number of bent sequences of length  $2^6$  including linear-based bent sequences and those constructed from four bent sequences of length  $2^4$ .

**Example 2** Let  $k = 3$  in Construction. Let  $\varphi$  be a nondegenerate linear transformation on  $V_5$ . Write  $\alpha_0 = (0, 0, 0, 0, 0)$ ,  $\alpha_1 = (0, 0, 0, 0, 1)$ ,  $\dots$ ,  $\alpha_{2^5-1} = (1, 1, 1, 1, 1)$ . Define  $\varphi$ , a nondegenerate linear transformation on  $V_5$  as follows

$$\begin{aligned} \varphi(\alpha_1) &= (0, 0, 0, 1, 1), & \varphi(\alpha_2) &= (0, 0, 1, 1, 0), & \varphi(\alpha_4) &= (0, 1, 1, 0, 0), \\ \varphi(\alpha_8) &= (1, 1, 0, 0, 0), & \varphi(\alpha_{16}) &= (1, 0, 0, 0, 0). \end{aligned}$$

Obviously,  $\{\alpha_1, \alpha_2, \alpha_4, \alpha_8, \alpha_{16}\}$  is a basis of  $V_5$ .

Write  $\varphi(\alpha_j) = \beta_j$  where  $j = 0, 1, \dots, 31$ . Hence we have

$$\begin{aligned} \beta_0 &= (0, 0, 0, 0, 0), & \beta_1 &= (0, 0, 0, 1, 1), & \beta_2 &= (0, 0, 1, 1, 0), & \beta_3 &= (0, 0, 1, 0, 1), \\ \beta_4 &= (0, 1, 1, 0, 0), & \beta_5 &= (0, 1, 1, 1, 1), & \beta_6 &= (0, 1, 0, 1, 0), & \beta_7 &= (0, 1, 0, 0, 1), \\ \beta_8 &= (1, 1, 0, 0, 0), & \beta_9 &= (1, 1, 0, 1, 1), & \beta_{10} &= (1, 1, 1, 1, 0), & \beta_{11} &= (1, 1, 1, 0, 1), \\ \beta_{12} &= (1, 0, 1, 0, 0), & \beta_{13} &= (1, 0, 1, 1, 1), & \beta_{14} &= (1, 0, 0, 1, 0), & \beta_{15} &= (1, 0, 0, 0, 1), \\ \beta_{16} &= (1, 0, 0, 0, 0), & \beta_{17} &= (1, 0, 0, 1, 1), & \beta_{18} &= (1, 0, 1, 1, 0), & \beta_{19} &= (1, 0, 1, 0, 1), \\ \beta_{20} &= (1, 1, 1, 0, 0), & \beta_{21} &= (1, 1, 1, 1, 1), & \beta_{22} &= (1, 1, 0, 1, 0), & \beta_{23} &= (1, 1, 0, 0, 1), \\ \beta_{24} &= (0, 1, 0, 0, 0), & \beta_{25} &= (0, 1, 0, 1, 1), & \beta_{26} &= (0, 1, 1, 1, 0), & \beta_{27} &= (0, 1, 1, 0, 1), \\ \beta_{28} &= (0, 0, 1, 0, 0), & \beta_{29} &= (0, 0, 1, 1, 1), & \beta_{30} &= (0, 0, 0, 1, 0), & \beta_{31} &= (0, 0, 0, 0, 1). \end{aligned}$$

Choose two bent sequences of length  $2^4$ :

$$\xi_1 = (+ + + + + - - + - + - + - - +) = (a_1, \dots, a_{16})$$

and

$$\xi_2 = (+ + + - + + + - + + + - - - - +) = (b_1, \dots, b_{16}).$$

Let the  $\beta_0$ -th, the  $\beta_1$ -th, ..., the  $\beta_{15}$ -th entries of  $\eta_1$  be  $a_1, a_2, \dots, a_{16}$  respectively and the  $\beta_{16}$ -th, the  $\beta_{17}$ -th, ..., the  $\beta_{31}$ -th entries of  $\eta_1$  be  $b_1, b_2, \dots, b_{16}$  respectively. We have now constructed  $\eta_1$ :

$$\eta_1 = (+ + - + - + + - + - - + + - + + + - + + - + - + - + - + +).$$

Also let the  $\beta_0$ -th, the  $\beta_1$ -th, ..., the  $\beta_{15}$ -th entries of  $\eta_2$  be  $a_1, a_2, \dots, a_{16}$  respectively and the  $\beta_{16}$ -th, the  $\beta_{17}$ -th, ..., the  $\beta_{31}$ -th entries of  $\eta_2$  be  $-b_1, -b_2, \dots, -b_{16}$  respectively. We have now constructed  $\eta_2$ :

$$\eta_2 = (+ - + + + + + - - - - + + - + - - + + - - + + - - - - + -).$$

Finally set  $\eta = (\eta_1, \eta_2)$ , by Lemma 6, this is a bent sequence of length of  $2^6$  by using  $\xi_1, \xi_2$  and  $\varphi$  in Construction 1.

Similarly we can construct another bent sequence by using  $\xi_1, \xi_2$  and  $\varphi$  in Construction 2. To do this set  $\eta'_1 = \eta_1$  and  $\eta'_2 = -\eta_2$ .  $\eta' = (\eta'_1, \eta'_2)$ , by Lemma 7, this is a bent sequence of length of  $2^6$  by using  $\xi_1, \xi_2$  and  $\varphi$  in Construction 2.

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