Semi-regular Sets of Matrices and Applications

Xian-Mo Zhang

Department of Computer Science
The University of Wollongong
Wollongong
NSW 2500, AUSTRALIA

Abstract

The concept of semi-regular sets of matrices was introducted by J. Seberry in " A new construction for Williamson-type matrices ", *Graphs and Combinatorics,* 2(1986), 81-87.
A regular s-set of matrices of order m. was first discovered by J. Seberry and A. L. Whiteman in "New Hadamard matrices and conference matrices obtained via Mathon's construction", *Graphs and Combinatorics*, 4(1988), 355-377. X. M. Zhang studied the product of these matrices and their applications in " Regular sets of matrices and applications " (to appear). In this paper, we prove that

(i) if there exist a semi-regular s-set of order m and a semi-regular t-set of order n then there exists a semi-regular s-set of order mn when $t = sm$,

(ii) if there exist Williamson type matrices of order n and a semi-regular $s(= 2n)$-set of matrices of order m then there exist Williamson type matrices of order nm,

(iii) if there exists a complex Hadamard matrix of order $2c$ and a semi-regular $s(= 2c)$-set of matrices of order m then there exists a complex Hadamard matrix of order $2cm$.

Williamson type matrices X_1, X_2, X_3, X_4 will be called nice if $X_1X_2^T + X_3X_4^T = 0$, perfect if $X_1X_2^T + X_3X_4^T = X_1X_4^T + X_2X_3^T = 0$, special if $X_1X_2^T + X_3X_4^T = X_1X_3^T + X_2X_4^T = X_1X_4^T + X_2X_3^T = 0$. We prove that

(iv) if there exist nice Williamson type matrices of order n and m then there exist Williamson type matrices of order nm,

(v) if there exist nice Williamson type matrices of order n and special Williamson matrices of order m then there exist nice Williamson type matrices of order nm,

(vi) if there exist special Williamson type matrices of order n and m then there exist special Williamson type matrices of order nm,

(vii) if there exist nice (perfect) Williamson type matrices of order n, where n is odd and $2n - 1$ is a prime power then there exist nice (perfect) Williamson matrices of order $n(2n - 1)^2$.

Let A_1, A_2, A_3, A_4 be type 1 (1 , -1)-matrices of order n will be called tight Williamson-like matrices if $\sum_{j=1}^{4} A_j A_j^T = 4nI_n$ and $A_1A_2^T + A_2A_1^T + A_3A_4^T + A_4A_3^T = 0$. We prove

(viii) if there exist tight Williamson-like matrices of order n and T-matrices of order t then there exists an Hadamard matrix of order $4tn$,

(ix) there exist two disjoint $W(2n, n)$ if there exist tight Williamson-like matrices of order n,

(x) if there exist tight Williamson-like matrices of order n and special Williamson type matrices of order m then there exist tight Williamson-like matrices of order nm,

(xi) if there exist tight Williamson-like matrices of order n (odd) and $2n-1$ is a prime power then there exist tight Williamson-like matrices of order $n(2n-1)^2$,

(xii) if there exist tight Williamson-like matrices of order n and $4n-1$ is a prime power then there exist tight Williamson-like matrices of order $n(4n-1)^2$.

1 Introduction and Basic Definitions

Definition 1 Suppose $Q_1, \ldots, Q_{2s}$ are $(1, -1)$ matrices of order m satisfying

$$Q_iQ_j^T = J, \quad i-j \neq 0,\ \pm s, \quad i,j \in \{1, \ldots, 2s\}, \tag{1}$$

$$Q_iQ_{i+s}^T = Q_{i+s}Q_i^T, \qquad i,j \in \{1, \ldots, 2s\}, \tag{2}$$

$$\sum_{i=1}^{2s} Q_iQ_i^T = 2smI_m. \tag{3}$$

Call $\{Q_1, \ldots, Q_{2s}\}$ a *semi-regular s-set of matrices* of order m.

Definition 2 Suppose $A_1, \ldots, A_s$ are $(1, -1)$ matrices of order m satisfying

$$A_iA_j = J, \quad i,j \in \{1, \ldots, s\}, \tag{4}$$

$$A_i^TA_j = A_j^TA_i = J, \quad i \neq j, \quad i,j \in \{1, \ldots, s\}, \tag{5}$$

$$\sum_{i=1}^{s}(A_iA_i^T + A_i^TA_i) = 2smI_m. \tag{6}$$

Call $\{A_1, \ldots, A_s\}$ a *regular s-set of matrices* of order m if (10), (11), (12) are satisfied (see [?], [8]).

Regular sets of matrices are special case of semi-regular sets of matrices. To show this, suppose $\{A_1, \dots, A_s\}$ is a regular s-set of matrices and set $Q_j = A_j$, $Q_{j+s} = A_j^T$, $j = 1, \dots, s$. Hence $\{Q_1, \dots, Q_{2s}\}$ is a semi-regular s-set of matrices. J. Seberry [7] gave semi-regular $\frac{1}{2}(q+1)$-set of matrices of order q^2, say $S_1, \dots, S_{q+1}$, satisfying $Q_i Q_j^T = Q_j Q_i^T = J_{q^2}$, $i \neq j$, where $q \equiv 3(mod\ 4)$, a prime power and (p+1)-set of matrices of order p^2, for $p \equiv 1(mod\ 4)$, a prime power. J. Seberry and A. L. Whiteman [8] proved that if $q \equiv 3 \pmod 4$ is a prime power there exists a regular $\frac{1}{2}(q+1)$-set of regular matrices of order q^2, say A_i, $i = 1, \dots, \frac{1}{2}(q+1)$ satisfying $A_i J = J A_i = qJ$. Semi-regular sets of matrices are not as convenient as regular sets of matrices because of the equality $Q_i Q_{i+s}^T = J$ and the non-commutativity of semi-regular s-sets of matrices.

Definition 3 Four $(1, -1)$ matrices X_1, X_2, X_3, X_4 of order n satisfying

$$X_1 X_1^T + X_2 X_2^T + X_3 X_3^T + X_4 X_4^T = 4n I_n$$

and

$$UV^T = VU^T,$$

where $U, V \in \{X_1, X_2, X_3, X_4\}$ will be called *Williamson type matrices* of order n [**?**]. Circulant and symmetric Williamson matrices will be called *Williamson matrices.*

Williamson and Williamson type matrices are discussed extensively by Baumert, Miyamoto, Seberry, Whiteman, Yamada and Yamamoto ([**?**], [1], [20], [21], [7], [12], [6], [14], [15], [5], [19], [9]).

Definition 4 Williamson type matrices (Williamson matrices) X_1, X_2, X_3, X_4 will be called *nice* if $X_1 X_2^T + X_3 X_4^T = 0$, *perfect* if $X_1 X_2^T + X_3 X_4^T = X_1 X_4^T + X_2 X_3^T = 0$, *special* if $X_1 X_2^T + X_3 X_4^T = X_1 X_3^T + X_2 X_4^T = X_1 X_4^T + X_2 X_3^T = 0$.

The concept of special Williamson matrices was introduced by Turyn [11], who found special Williamson matrices of order 9^j, for j a non-negative integer. Recently Xia [**?**] gave special Williamson type matrices of order $N = 9^i \prod_{j=1}^{t} q_j^{4r_j}$, where $q_j \equiv 3(mod\ 4)$, is a prime power, i, r_j are non-negative integers [**?**].

Definition 5 Type 1 (1, -1) matrices of order n A_1, A_2, A_3, A_4 will be called *tight Williamson-like matrices* if $\sum_{j=1}^{4} A_j A_j^T = 4n I_n$ and $A_1 A_2^T + A_2 A_1^T + A_3 A_4^T + A_4 A_3^T = 0$.

Definition 6 Let C be a $(1, -1, i, -i)$ matrix of order c satisfying $CC^* = cI_c$, where C^* is the Hermitian onjugate of C. We call C a *complex Hadamard matrix* of order c.

From [13], any complex Hadamard matrix has order 1 or order divisible by 2. Let $C = X + iY$, where X, Y consist of $1, -1, 0$ and $X \wedge Y = 0$ where $\wedge$ is the Hadamard product. Clearly, if C is a complex Hadamard matrix then $XX^T + YY^T = cI_c$, $XY^T = YX^T$.

Definition 7 Four type 1 (1 , -1)-matrices, say T_1, T_2, T_3, T_4 of order t will be called *T-matrices* if $T_i \wedge T_j = 0$ for $i \neq j$, where $\wedge$ is the Hadamard product, and $\sum_{j=1}^{4} T_j T_j^T = tI_t$.

Notation 1 For convenience, in this paper we write $N = 9^i \prod_{j=1}^{t} q_j^{4r_j}$, where $q_j \equiv 3(mod\, 4)$ is a prime power, i, r_j are non-negative integers.

Let $M = (M_{ij})$ and $N = (N_{gh})$ be orthogonal matrices with t^2 block M-structure (see [9]) of order tm and tn respectively, where M_{ij} is of order m $(i, j = 1, \ldots, t)$ and N_{gh} is of order n $(i, j = 1, 2, \ldots, t)$. We now define the the operation $\bigcirc$ as the following:

$$M \bigcirc N = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1t} \\ L_{21} & L_{22} & \cdots & L_{2t} \\ & & \cdots & \\ L_{t1} & L_{t2} & \cdots & L_{tt} \end{bmatrix}$$

where M_{ij}, N_{ij} and L_{ij} are of order of m, n and mn, respectively and

$$L_{ij} = M_{i1} \times N_{1j} + M_{i2} \times N_{2j} + \cdots + M_{it} \times N_{tj},$$

where $\times$ is Kronecker product, $i, j = 1, 2, \ldots, t$. We call this the *strong Kronecker* multiplication of two matrices (see [**?**]).

2 Existence of Semi-Regular Sets of Matrices

The following results are known:

Theorem 1 *Let both $p \equiv 1(mod\ 4)$ and $q \equiv 3(mod\ 4)$ be prime powers. Then*

(i) there exists a semi-regular (p+1)-set of matrices of order p^2 (J. Seberry [7]),

(ii) there exists a semi-regular $\frac{1}{2}(q+1)$-set of matrices of order q^2, say $S_1, \ldots, S_{q+1}$, satisfying $S_i S_j^T = J$, $i \neq j$, $i, j = 1, \ldots, q+1$ (J. Seberry [7], here we have changed the subscripts j to $j+1$),

(iii) there exists a regular $\frac{1}{2}(q+1)$-set of matrices of order q^2 (J. Seberry and A. L. Whiteman [8]).

Theorem 2 *If there exist a semi-regular s-set of matrices of order m and a semi-regular $t(=sm)$-set of matrices of order n then there exists a semi-regular s-set of matrices of order mn.*

Proof. Let $\{A_1 = (a^1_{ij}),\ A_2 = (a^2_{ij}), \ldots,\ A_{2s} = (a^{2s}_{ij})\}$ be the semi-regular s-set of matrices of order m and $\{\ B_1,\ B_2,\ \ldots\ ,\ B_{2t}\ \}$ be the semi-regular t-set of matrices of order of n.
Define $C_i = (c^i_{kj}) = (a^i_{kj} B_{(i-1)m+j+k-1})$, $i = 1, \ldots, 2s$ so that

$$C_i = \begin{bmatrix} a^i_{11}B_{(i-1)m+1} & a^i_{12}B_{(i-1)m+2} & \cdots & a^i_{1m}B_{im} \\ a^i_{21}B_{(i-1)m+2} & a^i_{22}B_{(i-1)m+3} & \cdots & a^i_{2m}B_{(i-1)m+1} \\ & & \vdots & \\ a^i_{m1}B_{im} & a^i_{m2}B_{(i-1)m+1} & \cdots & a^i_{mm}B_{im-1} \end{bmatrix}.$$

For any $i,j,\ i-j \neq 0,\ \pm s$, there exist no $B_u,\ B_v$ such that $u - v = \pm t$, B_u in C_i, B_v in C_j. Thus $C_iC_j = J_m \times J_n = J_{mn}$, for $i,j,\ i-j\ \neq 0,\ \pm s$. On the other hand, for a fixed i, write $C_iC^T_{i+s} = (D_{uv})$, where D_{uv} is of order n, $u,v = 1,\ldots,m$. Obviously, $D_{uv} = J_n$, for $u \neq v$. Note $D_{uu} = \sum_{k=1}^m a^i_{uk}a^{i+s}_{vk}B_{(i-1)m+k}B^T_{(i+s-1)m+k}$. Since $B_kB^T_{k+s} = B_{k+s}B^T_k$, $D^T_{uu} = D_{uu}$. Thus $C_iC^T_{i+s}$ is symmetric i.e. $C_iC^T_{i+s} = C_{i+s}C^T_i$.
To show

$$\sum_{i=1}^{2s} C_iC_i^T = 2smnI_{mn}, \tag{7}$$

note that $(a^i_{kj})^2 = 1$ so the diagonal element of $C_iC_i^T$ is $\sum_{j=1}^m B_{(i-1)m+j}B^T_{(i-1)m+j}$ and hence the diagonal element of $\sum_{i=1}^{2s} C_iC_i^T$ is

$$\sum_{j=1}^{2sm} B_jB_j^T = \sum_{j=1}^{2t} B_jB_j^T = 2tnI_n = 2smnI_n.$$

The off-diagonal elements of $C_iC_i^T$ are given by

$$\sum_{j=1}^{m}(a^i_{hj}a^i_{kj}B_{(i-1)m+j+h-1}B^T_{(i-1)m+j+k-1}),\ h \neq k$$

$$= \sum_{j=1}^{m} a^i_{hj}a^i_{kj}J.$$

So the off-diagonal element of $\sum_{i=1}^{s} C_iC_i^T$ is zero, using

$$\sum_{i=1}^{s}\sum_{j=1}^{m} a^i_{hj}a^i_{kj}J = 0.$$

□

Corollary 1 *Let both $p \equiv 1(mod\, 4)$ and $q \equiv 3(mod\, 4)$ be prime powers.*

(i) if $(p+1)p^2 - 1$ is a prime power then there exists a semi-regular (p+1)-set of matrices of $p^2((p+1)p^2-1)^2$,

(ii) if $2(p+1)p^2 - 1$ is a prime power then there exists a semi-regular (p+1)-set of matrices of $p^2(2(p+1)p^2-1)^2$,

(iii) if $(q+1)q^2 - 1$ is a prime power then there exists a regular $\frac{1}{2}(q+1)$-set of matrices of $q^2((q+1)q^2-1)^2$,

(iv) if $\frac{1}{2}(q+1)q^2 - 1$ is a prime power then there exists a semi-regular $\frac{1}{2}(q+1)$ -set of matrices of $q^2(\frac{1}{2}(q+1)q^2-1)^2$.

Proof. (i) By Theorem 1 there exists a semi-regular (p+1)-set of matrices of order p^2. Since $(p+1)p^2 - 1 \equiv 1(mod\, 4)$, by Theorem 1, there exists a semi-regular $(p+1)p^2$-set of matrices of $((p+1)p^2-1)^2$. Using Theorem 2, there exists a (p+1)-set of matrices of $p^2((p+1)p^2-1)^2$.
(ii) By Theorem 1 there exists a semi-regular (p+1)-set of matrices of order p^2. Since $2(p+1)p^2 - 1 \equiv 3(mod\, 4)$, By Theorem 1 there exists a semi-regular $(p+1)p^2$-set of matrices of $(2(p+1)p^2-1)^2$. Using Theorem 2, there exists a (p+1)-set of matrices of $p^2(2(p+1)p^2-1)^2$.
(iii) This is Corollary 2, [**?**].
(iv) By Theorem 1 there exists a semi-regular $\frac{1}{2}(q+1)$-set of matrices of order q^2. Case 1, $q \equiv 3(mod\, 8)$. Then $\frac{1}{2}(q+1)q^2 - 1 \equiv 1(mod\, 4)$. By Theorem 1 there exists a semi-regular $\frac{1}{2}(q+1)q^2$-set of $(\frac{1}{2}(q+1)q^2-1)^2$. By Theorem 2 there exists a semi-regular $\frac{1}{2}(q+1)$-set of matrices of $q^2(\frac{1}{2}(q+1)q^2-1)^2$. Case 2, $q \equiv 7(mod\, 8)$. This is Corollary 5 [**?**], we still have a semi-regular $\frac{1}{2}(q+1)$-set of matrices of $q^2(\frac{1}{2}(q+1)q^2-1)^2$. □

3 Williamson Type Matrices and Complex Hadamard Matrices

We find new constructions for Williamson type matrices not given by Miyamoto [5] or Seberry and Yamada [**?**], [9]. This theorem differs from that of Seberry [7] as it does not need regular sets of regular matrices.

Theorem 3 *If there exist Williamson type matrices of order n and a semi-regular $s(=2n)$-set of matrices of order m then there exist Williamson type matrices of order nm.*

Proof. Let $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, $D = (d_{ij})$ be the Williamson type matrices of order n and $R_1, \ldots, R_{2s}$ be the semi-regular s-set of matrices of order m. Set $E = (a_{ij}R_{j+i-1})$, $F = (b_{ij}R_{n+j+i-1})$, $G = (c_{ij}R_{2n+j+i-1})$, $H = (d_{ij}R_{3n+j+i-1})$, where $i, j = 1, \ldots, n$ and the subscripts $j+i-1$ are reduced modulo n. By the same reasoning as in the proof for Theorem 4 [7], E, F, G, H are Williamson type matrices of order nm. $\Box$

Corollary 2 *If n is the odd order of Williamson type matrices and $2n-1$ is a prime power then there exist Williamson type matrices of order $n(2n-1)^2$.*

Proof. Since n is odd, $2n-1 \equiv 1 (mod\, 4)$. By Theorem 1 there exists a semi-regular $2n$-set of matrices of $(2n-1)^2$. By Theorem 3 we have Williamson type matrices of order $n(2n-1)^2$. $\Box$

Corollary 3 *(i) There exist Williamson type matrices of order $9^k(2\cdot 9^k - 1)^2$ if $2\cdot 9^k - 1$ is a prime power, where k is a non-negative interger,*

(ii) there exist Williamson type matrices of order $7\cdot 3^k(14\cdot 3^k - 1)^2$ if $14\cdot 3^k - 1$ is a prime power, where k is a non-negative interger.

Proof. From the Index of [**?**], there exist Williamson type matrices of order of 9^k and $7\cdot 3^k$, where $k = 0, 1, \ldots$. Using Corollary 2, the corollary is established. $\Box$

Theorem 4 *If there exist a complex Hadamard matrix of order $2c$ and a semi-regular $s(=2c)$-set of matrices of order m then there exists a complex Hadamard matrix of order $2cm$.*

Proof. Let $\{ A_1, \ldots, A_{2s} \}$ be the semi-regular $s(=2c)$-set of matrices of order m and $C = X+iY$ be the complex Hadamard matrix of order $2c$, where both X and Y are $(1,-1)$ matrices satisfying $X \wedge Y = 0$, $XX^T + YY^T = 2cI_{2c}$, $XY^T = YX^T$. Let $P = X+Y$ and $Q = X-Y$. Then both P and Q are $(1,-1)$ matrices of order $2c$ and $PP^T + QQ^T = 4cI_{2c}$, $PQ^T = QP^T$. Let $P = (p_{ij})$ and $Q = (q_{ij})$, $i, j = 1, \ldots, 2c$. Set $E = (p_{ij}A_{i+j-1})$ and $F = (q_{ij}A_{s+i+j-1})$, where $i, j = 1, \ldots, s$ and the subscripts $i+j-1$ are reduced modulo s, $1, \ldots, s = 2c$. Clearly, both E and F are $(1,-1)$ matrices of order $2cm$, since both P and Q are $(1,-1)$ matrices of order $2c$. We now prove $EE^T + FF^T = 4cmI_{2cm}$. Write $E = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{bmatrix}$ and $F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix}$, where E_i and F_i are matrices of order $m \times sm$. Note

$$E_iE_i^T + F_iF_i^T = \sum_{j=1}^{s}(p_{ij}p_{ij}A_{i+j-1}A_{i+j-1}^T + q_{ij}q_{ij}A_{s+i+j-1}A_{s+i+j-1}^T)$$

$$= \sum_{j=1}^{s}(A_jA_j^T + A_jA_j^T) = \sum_{j=1}^{2s} A_jA_j^T = 2smI_m.$$

On the other hand, if $i \neq k$,

$$E_iE_k^T + F_iF_k^T = \sum_{j=1}^{s}(p_{ij}p_{kj}A_{i+j-1}A_{k+j-1}^T + q_{ij}q_{kj}A_{s+i+j-1}A_{s+k+j-1}^T)$$

$$= \sum_{j=1}^{s}(p_{ij}p_{kj} + q_{ij}q_{kj})J_m = 0.$$

Thus

$$EE^T + FF^T = 2smI_{sm} = 4cmI_{2cm}.$$

We now prove $EF^T = FE^T$. Write $EF^T = (D_{ij})$, where D_{ij} is of order m, $i, j = 1, \ldots, 2c$. Note $D_{ij} = \sum_{k=1}^{2c} p_{ik}q_{jk}A_{i+k-1}A_{s+j+k-1}^T$. For $i \neq j$, $D_{ij} = \sum_{k=1}^{2c} p_{ik}q_{jk}J_m$. Since $PQ^T = QP^T$, $D_{ij}^T = D_{ji}$, $i \neq j$. Note $D_{ii} = \sum_{k=1}^{2c} p_{ik}q_{ik}A_{i+k-1}A_{s+i+k-1}^T$. From (8), Definition 1, $D_{ii}^T = D_{ii}$. Thus EF^T is symmetric i.e. $EF^T = FE^T$. Finally, Set $U = \frac{1}{2}(E+F)$ and $V = \frac{1}{2}(E-F)$. Note both E and F are $(1,-1)$ matrices of order $2cm$ then both U and V are $(1,-1,0)$ matrices of order $2cm$ satisfying $U \wedge V = 0$, $UU^T + VV^T = \frac{1}{2}(EE^T + FF^T) = 2cmI_{2cm}$. Since $EF^T = FE^T$, $UV^T = VU^T$. Thus $U + iV$ is a complex Hadamard matrix of order $2cm$. □

Corollary 4 *If both $p \equiv 1 \pmod 4$ and $p^j(p+1)-1$ are prime powers then there exists a complex Hadamard matrix of order $p^j(p+1)(p^j(p+1)-1)^2$, where j is a positive integer.*

Proof. Obviously, $p^j(p+1) - 1 \equiv 1 \pmod 4$. By Theorem 1 there exists a regular $p^j(p+1)$-set of matrices of order $(p^j(p+1)-1)^2$. From Corollary 18, [4], there exists a complex Hadamard matrix of order $p^j(p+1)$. Using Theorem 3, we have a $p^j(p+1)(p^j(p+1)-1)^2$. □

4 Product and New Construction of Williamson type Matrices with Additional Properties

Part (iii) of the next theorem was known to Turyn [11] but we put here for completeness.

Theorem 5 *(i) If there exist nice Williamson type matrices of order n and m then there exist Williamson type matrices of order nm,*

(ii) *if there exist nice Williamson type matrices of order n and special Williamson matrices of order m then there exist nice Williamson type matrices of order nm,*

(iii) *if there exist special Williamson type matrices of order n and m then there exist special Williamson type matrices of order nm.*

Proof. Let X_1, X_2, X_3, X_4 be nice Williamson type matrices of order n and Y_1, Y_2, Y_3, Y_4 be nice Williamson type matrices of order m. Set

$$Z_1 = \frac{1}{2}(X_1+X_2)\times Y_1+\frac{1}{2}(X_1-X_2)\times Y_2,\ Z_2 = \frac{1}{2}(X_1+X_2)\times Y_3+\frac{1}{2}(X_1-X_2)\times Y_4,$$

$$Z_3 = \frac{1}{2}(X_3+X_4)\times Y_1+\frac{1}{2}(X_3-X_4)\times Y_2,\ Z_4 = \frac{1}{2}(X_3+X_4)\times Y_3+\frac{1}{2}(X_3-X_4)\times Y_4.$$

Then Z_1, Z_2, Z_3, Z_4 are $(1,-1)$ matrices of order nm. Note

$$\begin{aligned}
Z_1Z_1^T &= \tfrac{1}{4}(X_1+X_2)(X_1+X_2)^T \times Y_1Y_1^T + \tfrac{1}{4}(X_1-X_2)(X_1-X_2)^T \times Y_2Y_2^T \\
&+ \tfrac{1}{2}(X_1+X_2)(X_1-X_2)^T \times Y_1Y_2^T, \\
Z_2Z_2^T &= \tfrac{1}{4}(X_1+X_2)(X_1+X_2)^T \times Y_3Y_3^T + \tfrac{1}{4}(X_1-X_2)(X_1-X_2)^T \times Y_4Y_4^T \\
&+ \tfrac{1}{2}(X_1+X_2)(X_1-X_2)^T \times Y_3Y_4^T, \\
Z_3Z_3^T &= \tfrac{1}{4}(X_3+X_4)(X_3+X_4)^T \times Y_1Y_1^T + \tfrac{1}{4}(X_3-X_4)(X_3-X_4)^T \times Y_2Y_2^T \\
&+ \tfrac{1}{2}(X_3+X_4)(X_3-X_4)^T \times Y_1Y_2^T, \\
Z_4Z_4^T &= \tfrac{1}{4}(X_3+X_4)(X_3+X_4)^T \times Y_3Y_3^T + \tfrac{1}{4}(X_3-X_4)(X_3-X_4)^T \times Y_4Y_4^T \\
&+ \tfrac{1}{2}(X_3+X_4)(X_3-X_4)^T \times Y_3Y_4^T.
\end{aligned}$$

It is easy to check that

$$Z_1Z_1^T + Z_2Z_2^T + Z_3Z_3^T + Z_4Z_4^T$$

$$= \frac{1}{4}(X_1X_1^T+X_2X_2^T+X_3X_3^T+X_4X_4^T)\times(Y_1Y_1^T+Y_2Y_2^T+Y_3Y_3^T+Y_4Y_4^T) = 4nmI_{nm}.$$

Obviously, $Z_iZ_j^T = Z_jZ_i^T$, for $i,j = 1,2,3,4$. Thus, Z_1, Z_2, Z_3, Z_4 are Williamson type matrices of order nm.
In particular, let X_1, X_2, X_3, X_4 be nice Williamson type matrices of order n and Y_1, Y_2, Y_3, Y_4 be special Williamson type matrices of order m. Note

$$Z_1Z_2^T = \frac{1}{4}(X_1+X_2)(X_1+X_2)^T \times Y_1Y_3^T + \frac{1}{4}(X_1-X_2)(X_1-X_2)^T \times Y_2Y_4^T$$

$$+\frac{1}{4}(X_1+X_2)(X_1-X_2)^T \times Y_1Y_4^T + \frac{1}{4}(X_1-X_2)(X_1+X_2)^T \times Y_2Y_3^T,$$

where

$$(X_1+X_2)(X_1-X_2)^T \times Y_1Y_4^T + (X_1-X_2)(X_1+X_2)^T \times Y_2Y_3^T$$
$$= (X_1X_1^T - X_2X_2^T) \times (Y_1Y_4^T + Y_2Y_3^T) = 0.$$

Then

$$Z_1Z_2^T = \frac{1}{4}(X_1+X_2)(X_1+X_2)^T \times Y_1Y_3^T + \frac{1}{4}(X_1-X_2)(X_1-X_2)^T \times Y_2Y_4^T.$$

Similarly,

$$Z_3Z_4^T = \frac{1}{4}(X_3+X_4)(X_3+X_4)^T \times Y_1Y_3^T + \frac{1}{4}(X_3-X_4)(X_3-X_4)^T \times Y_2Y_4^T.$$

Hence

$$Z_1Z_2^T + Z_3Z_4^T = \frac{1}{4}(X_1X_1^T + X_2X_2^T + X_3X_3^T + X_4X_4^T) \times (Y_1Y_3^T + Y_2Y_4^T) = 0.$$

We have now proved Z_1, Z_2, Z_3, Z_4 are nice Williamson type matrices of order nm.
Further suppose X_1, X_2, X_3, X_4 are special Williamson type matrices of order n and Y_1, Y_2, Y_3, Y_4 are special Williamson type matrices of order m.

$$Z_1Z_3^T = \frac{1}{4}(X_1+X_2)(X_3+X_4)^T \times Y_1Y_1^T + \frac{1}{4}(X_1-X_2)(X_3-X_4)^T \times Y_2Y_2^T$$
$$+\frac{1}{4}(X_1+X_2)(X_3-X_4)^T \times Y_1Y_2^T + \frac{1}{4}(X_1-X_2)(X_3+X_4)^T \times Y_2Y_1^T.$$

Note

$$(X_1+X_2)(X_3+X_4)^T = (X_1-X_2)(X_3-X_4)^T = 0,$$

then

$$Z_1Z_3^T = \frac{1}{4}(X_1+X_2)(X_3-X_4)^T \times Y_1Y_2^T + \frac{1}{4}(X_1-X_2)(X_3+X_4)^T \times Y_2Y_1^T.$$

Similarly,

$$Z_2Z_4^T = \frac{1}{4}(X_1+X_2)(X_3-X_4)^T \times Y_3Y_4^T + \frac{1}{4}(X_1-X_2)(X_3+X_4)^T \times Y_4Y_3^T.$$

Clearly, $Z_1Z_3^T + Z_2Z_4^T = 0$. Finally, by the same reasoning for $Z_1Z_3^T$ and $Z_2Z_4^T$, we have

$$Z_1Z_4^T = \frac{1}{4}(X_1+X_2)(X_3-X_4)^T \times Y_1Y_4^T + \frac{1}{4}(X_1-X_2)(X_3+X_4)^T \times Y_2Y_3^T$$

and

$$Z_2Z_3^T = \frac{1}{4}(X_1+X_2)(X_3-X_4)^T \times Y_3Y_2^T + \frac{1}{4}(X_1-X_2)(X_3+X_4)^T \times Y_4Y_1^T.$$

Clearly $Z_1Z_4^T + Z_2Z_3^T = 0$. Thus Z_1, Z_2, Z_3, Z_4 are special Williamson type matrices of order nm. $\square$

Let $q \equiv 1 (mod\, 4)$ be a prime power and $n = \frac{1}{2}(1+q)$. By Theorem 1 there exists a semi-regular $2n = (p+1)$-set of matrices of order p^2, i.e. $4n$ (1 , -1)-matrices $Q_1, \ldots, Q_{4n}$ satisfying

$$Q_i Q_j^T = J_{p^2},\ \textit{if}\ i-j \neq \pm 2n, 0,\ Q_i Q_{i+2n}^T = Q_{i+2n} Q_i^T$$

and

$$\sum_{j=1}^{4n} Q_j Q_j^T = 4q^2(1+p) I_{p^2}.$$

Suppose there exist Williamson type matrices of order n, say $A = (a_{ij})$, $B = (b_{ij})$, $C = (b_{ij})$, $D = (d_{ij})$. Set

$$E = (a_{ij} Q_{1+j-i}),\ F = (b_{ij} Q_{n+1+j-i}),\ G = (c_{ij} Q_{2n+1+j-i}),\ H = (d_{ij} Q_{3n+1+j-i}),$$

where $1+j-i$ is the reduced modulo n to the residue class $\{1, \ldots, n\}$. By the same reasoning as in the proof of Theorem 4 [7], E, F, G, H are Williamson type matrices of order nq^2. Further suppose $AB^T + CD^T = 0$ i.e. A, B, C, D are nice Williamson matrices of order n. Write $EF^T = (X_{ij})$, $GH^T = (Y_{ij})$, where X_{ij}, Y_{ij} are of order q^2, $i, j = 1, \ldots n$. Note

$$X_{ij} = \sum_{k=1}^{n} a_{ik} Q_{1+k-i} b_{jk} Q_{n+1+k-j}^T = \sum_{k=1}^{n} a_{ik} b_{jk} J_{q^2},$$

since $(n+1+k-j) - (1+k-i) \neq 0,\ 2n$. Similarly,

$$Y_{ij} = \sum_{k=1}^{n} c_{ik} Q_{2n+1+k-i} b_{jk} Q_{3n+1+k-j}^T = \sum_{k=1}^{n} c_{ik} d_{jk} J_{q2},$$

since $(3n+1+k-j) - (2n+1+k-i) \neq 0, 2n$. Note $AB^T + CD^T = 0$, then $X_{ij} + Y_{ij} = 0$. Thus $EF^T + GH^T = 0$. Similarly, if $AD^T + BC^T = 0$ then $EH^T + FG^T = 0$. Note if n is odd, then $2n-1 \equiv 1(mod\, 4)$. Hence we have proved

Theorem 6 *If there exist nice (perfect) Williamson type matrices of order n, where n is odd and $2n-1$ is a prime power then there exist nice (perfect) Williamson type matrices of order $n(2n-1)^2$.*

Corollary 5 *If there exist nice Williamson type matrices of order n and m then there exist Williamson type matrices of order nmN, where N was defined in Notation 1.*

Proof. From [?], there exist special Williamson type matrices of order N. By Theorem 5 there exist nice Williamson type matrices of order mN hence Williamson type matrices of order nmN. □

Corollary 6 *Let N, N_1 and N_2 be three products of the kind defined by Notation 1. If $2N-1$ is a prime power then there exist*

(i) perfect Williamson type matrices of order $N(2N-1)^2$,

(ii) nice Williamson type matrices of order $N(2N-1)^2N_1$,

(iii) Williamson type matrices of order $NN_1N_2(2N-1)^2(2N_2-1)^2$, if $2N_1-1$ is a prime power.

Proof. (i), (ii) and (iii) hold by Theorem 6, Theorem 5 and Corollary 5 respectively. □

By Corollary 6 there exist perfect Williamson type matrices of order $9 \cdot 17^2$, nice Williamson type matrices of order $9 \cdot 17^2 N$ and Williamson type matrices of order $9^2 \cdot 17^4 N$.

5 Tight Williamson-like Matrices and Application

Some tight Williamson-like matrices were found by Xia [18]. For example, from [16], we construct cyclic tight Williamson-like matrices of order 5 and 13 with first rows

$+-++-$, $++-++$, $--++-$, $++++-$ and

$++---+--++-++$, $--+++-+++++-+$,
$+--+-+++---+-$, $+-+++++++--+-$ respectively.

From [16] we construct type 1 tight Williamson-like matrices of order 25. Any element in the abelian group $Z_5 \oplus Z_5$ can be expressed as (a,b), where a, $b \in Z_5$, and the additive addition in $Z_5 \oplus Z_5$ can be defined as $(a,b)+(c,d)=(a+b,c+d)$. Set

$$\begin{aligned}
S_1 &= \{(0,0), (0,1), (1,2), (3,3), (0,3), (4,4), (3,4), (2,0), (2,2), (1,0), (1,4), (0,2), (3,0)\}, \\
S_2 &= \{(0,1), (4,0), (3,1), (4,4), (0,4), (4,2), (1,0), (1,1), (3,2)\}, \\
S_3 &= \{(1,2), (3,3), (1,3), (4,1), (3,4), (2,0), (2,3), (4,3), (1,4), (0,2), (2,4), (2,1)\}, \\
S_4 &= \{(3,3), (4,1), (0,3), (2,0), (4,3), (2,2), (0,2), (2,1), (3,0)\}.
\end{aligned}$$

Hence the type 1 (1, -1) incidence matrices of S_1, S_2, S_3, S_4 form the tight Williamson-like matrices of order 25.
Tight Williamson-like matrices are not Williamson type matrices but they are Goethals-Seidel or Wallis-Whiteman matrices (see [10]) with cross correlation types of properties (see Definition 4). Besides forming Hadamard matrices of Goethals-Seidel or Wallis-Whiteman type (see [10]), tight Williamson-like matrices can be used to form Hadamard matrices in the special array given in the next lemma.

Lemma 1 *If there exist tight Williamson-like matrices of order n then there exists an Hadamard matrix of order $4n$.*

Proof. Let A_1, A_2, A_3, A_4 be the tight Williamson-like matrices of order n. Set

$$H = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2 & A_1 & A_4 & A_3 \\ A_3^T & A_4^T & -A_1^T & -A_2^T \\ A_4^T & A_3^T & -A_2^T & -A_1^T \end{bmatrix}.$$

Hence H is an Hadamard matrices with 4×4 type 1 blocks. □

Lemma 2 *If there exist tight Williamson-like matrices of order n and T-matrices of order t then there exists an Hadamard matrix of order $4tn$.*

Proof. Let A_1, A_2, A_3, A_4 be the tight Williamson-like matrices of order n. Let T_1, T_2, T_3, T_4 be the T-matrices of order t. Write

$$\begin{aligned} E_1 &= T_1 \times A_1 + T_2 \times A_2 + T_3 \times A_3^T + T_4 \times A_4^T, \\ E_2 &= T_1 \times A_2 + T_2 \times A_1 + T_3 \times A_4^T + T_4 \times A_3^T, \\ E_3 &= T_1 \times A_3 + T_2 \times A_4 - T_3 \times A_1^T - T_2 \times A_2^T, \\ E_4 &= T_1 \times A_4 + T_2 \times A_3 - T_3 \times A_2^T - T_1 \times A_2^T. \end{aligned}$$

Clearly, each E_j is a (1, -1)-matrix. It is easy to check that $\sum_{j=1}^{4} E_j E_j^T = 4tnI_{tn}$. Note the E_j are of type 1, hence we can construct an Hadamard matrix of order $4tn$ by using Theorem 3 [10]. □

Lemma 3 *There exist two disjoint $W(2n, n)$ if there exist tight Williamson-like matrices of order n.*

Proof. Let A_1, A_2, A_3, A_4 be the tight Williamson-like matrices of order n. Set $P = \begin{bmatrix} A+B & C+D \\ C^T + D^T & -A^T - B^T \end{bmatrix}$ and $Q = \begin{bmatrix} A-B & C-D \\ C^T - D^T & -A^T + B^T \end{bmatrix}$. Then P and Q are the required two disjoint $W(2n, n)$. □

By Corollary 2.11 [3], a $W(2n, n)$, where n is odd, only exists when n is a sum of two squares. Hence, as poimted out in [17] we have

Lemma 4 *If there exist tight Williamson-like matrices of order n, odd, then n is a sum of two squares.*

Two disjoint $W(2n, n)$ are often used for constructing Hadamard matrices (see Craigen [2], [?]). We now construct new tight Williamson-like matrices from those known.

Lemma 5 *(Xia [18]) If there exist tight Williamson-like matrices of order n and special Williamson matrices of order m then there exist tight Williamson-like matrices of order nm.*

Lemma 6 *If there exist tight Williamson-like matrices of order n (odd) and $2n-1$ is a prime power then there exist tight Williamson-like matrices of order $n(2n-1)^2$.*

Proof. Let $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, $D = (d_{ij})$ be the tight Williamson-like matrices of order n. Note $2n-1 \equiv 1(mod\ 4)$, since n is odd. By Theorem 1 there exists a semi-regular $(2n)$-set of matrices of order $(2n-1)^2$, say $R_1, \ldots, R_{4n}$. Set $E = (a_{ij}R_{j+i-1})$, $F = (b_{ij}R_{n+j+i-1})$, $G = (c_{ij}R_{2n+j+i-1})$, $H = (d_{ij}R_{3n+j+i-1})$, where $i,j = 1,\ldots,n$ and the subscripts $j+i-1$ are reduced modulo n. By the same reasoning as in the proof for Theorem 4 [7], E,F,G,H satisfy

$$EE^T + FF^T + GG^T + HH^T = 4n(2n-1)^2 I_{n(2n-1)^2}.$$

On the other hand,

$$EF^T + FE^T + GH^T + HG^T = (AB^T + BA^T + CD^T + DC^T) \times J_{(2n-1)^2} = 0.$$

Thus E,F,G,H are tight Williamson-like matrices of order $n(2n-1)^2$. □

Lemma 7 *If there exist tight Williamson-like matrices of order n and $4n-1$ is a prime power then there exist tight Williamson-like matrices of order $n(4n-1)^2$.*

Proof. Let $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, $D = (d_{ij})$ be the tight Williamson-like matrices of order n. Note $4n-1 \equiv 3(mod\ 4)$. By Theorem 1 there exists a regular $(2n)$-set of matrices of order $(4n-1)^2$, say $A_1, \ldots, A_{4n}$. Set $E = (a_{ij}A_{j+i-1})$, $F = (b_{ij}A_{n+j+i-1})$, $G = (c_{ij}A^T_{j+i-1})$, $H = (d_{ij}A^T_{n+j+i-1})$, where $i,j = 1,\ldots,n$ and the subscripts $j+i-1$ are reduced modulo n. By the same reasoning as in the proof for Theorem 4 [7], E,F,G,H satisfy

$$EE^T + FF^T + GG^T + HH^T = 4n(4n-1)^2 I_{n(4n-1)^2}.$$

On the other hand,

$$EF^T + FE^T + GH^T + HG^T = (AB^T + BA^T + CD^T + DC^T) \times J_{(4n-1)^2} = 0.$$

Thus E,F,G,H are tight Williamson-like matrices of order $n(4n-1)^2$. □

Corollary 7 *There exist tight Williamson-like matrices of order* $5N$, $13N$, $25N$, $13 \cdot 5^4 N$, $5 \cdot 19^2 N$.

Proof. As the above, there exist tight Williamson-like matrices of order 5, 13, 25. By Lemma 6 there exist tight Williamson-like matrices of order $13 \cdot 5^4$. By Lemma 7 there exist tight Williamson-like matrices of order $5 \cdot 19^2$. Using Lemma 5, we have established the corollary. □

Corollary 8 *There exist Hadamard matrices of orders* $4n$, *where* $n = 5tN$, $13tN$, $25tN$, $13 \cdot 5^4 tN$, $5 \cdot 19^2 tN$, *where* t *is the order of T-matrices.*

Proof. Using Corollary 7 and Lemma 2. □

Corollary 9 *There exist two disjoint* $W(2n, n)$, *where* $n = 5N$, $13N$, $25N$, $13 \cdot 5^4 N$, $5 \cdot 19^2 N$.

Proof. Using Corollary 7 and Lemma 3. □

The following table shows the existence of tight Williamson-like matrices of odd order < 60. Tight Williamson-like matrices for odd orders, n, can only exist for $n \equiv 1 (mod\ 4)$, where the decomposition of n into prime powers contains no factor $p \equiv 3 (mod\ 4)$ raised to an odd power. Hence the following list contains only those n which exist or could possibly exist.

order	construction
5	[18], see Section 5
9^t	[11], since special Williamson type are tight Williamson-like matrices
13	[18], see Section 5
17	unknown
25	[18], see Section 5
29	unknown
37	unknown
41	unknown
45	[18], see Lemma 5
49	uncertain
53	unknown

Acknowlegement: I wish to thank Professor Jennifer Seberry for her help and encouragement.

References

[1] Baumert, L. D., and Hall, Jr., M. Hadamard matrices of Williamson type. *Math. Comp. 19* (1965), 442–447.

[2] Craigen, R. Constructing Hadamard matrices with orthogonal pairs. *Ars Combinatoria 33* (1992), 57–64.

[3] Geramita, A. V., and Seberry, J. *Orthogonal Designs: Quadratic Forms and Hadamard Matrices.* Marcel Dekker, New York-Basel, 1979.

[4] Kharaghani, H., and Seberry, J. Regular complex Hadamard matrices. *Congress. Num. 24* (1990), 149–151.

[5] Miyamoto, M. A construction for Hadamard matrices. *Journal of Combinatorial Theory Ser. A, 57* (1991), 86–108.

[6] Seberry, J. Some matrices of Williamson-type. *Utilitas Math. 4* (1973), 147–154.

[7] Seberry, J. A new construction for Williamson-type matrices. *Graphs and Combinatorics 2* (1986), 81–87.

[8] Seberry, J., and Whiteman, A. L. New Hadamard matrices and conference matrices obtained via Mathon's construction. *Graphs and Combinatorics 4* (1988), 355–377.

[9] Seberry, J., and Yamada, M. On the products of Hadamard matrices, Williamson matrices and other orthogonal matrices using M-structures. *JCMCC 7* (1990), 97–137.

[10] Seberry-Wallis, J. On Hadamard matrices. *Journal of Combinatorial Theory Ser. A, 18* (1975), 149–164.

[11] Turyn, R. J. A special class of Williamson matrices and difference sets. *Journal of Combinatorial Theory Ser. A, 36* (1984), 111–115.

[12] Wallis, J. S. Construction of Williamson type matrices. *Linear and Multilinear Algebra 3* (1975), 197–207.

[13] Wallis, W. D., Street, A. P., and Wallis, J. S. *Combinatorics: Room Squares, sum-free sets, Hadamard Matrices*, vol. 292 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[14] Whiteman, A. L. An infinite family of Hadamard matrices of Williamson type. *Journal of Combinatorial Theory Ser. A. 14* (1973), 334–340.

[15] Whiteman, A. L. Hadamard matrices of Williamson type. *J. Austral. Math. Soc. 21* (1976), 481–486.

[16] Xia, M. Y. Some supplementary difference sets and hadamard matrices. *Acta. Math. Sci. 4 (1)* (1984), 81–92.

[17] Xia, M. Y. On construction of some supplementary difference sets and others. *Journal of Central China Normal University (Natural Sciences), Monograph of Mathematics No. 1* (1989), 204–208.

[18] Xia, M. Y. Hadamard matrices. *Combinatorial designs and applications, Lecture Notes in Pure and Appl. Math., Dekker, New York 126* (1990), 179–181.

[19] Yamada, M. On the Williamson matrices of Turyn's type and and type j. *Comment. Math. Univ. San Pauli 31* (1982), 71–73.

[20] Yamamoto, K. On a generalized Williamson equation. *Colloq. Math. Janos Bolyai 37* (1981), 839–850.

[21] Yamamoto, K., and Yamada, M. Williamson matrices of Turyn's type and Gauss sums. *J. Math. Soc. Japan 37* (1985), 703–717.