Regular Sets of Matrices and Applications

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Abstract

Suppose $A_1, \cdots, A_s$ are $(1, -1)$ matrices of order m satisfying

$$A_iA_j = J, \quad i, j \in \{1, \cdots, s\}, \tag{1}$$

$$A_i^TA_j = A_j^TA_i = J, \quad i \neq j, \quad i, j \in \{1, \cdots, s\}, \tag{2}$$

$$\sum_{i=1}^{s}(A_iA_i^T + A_i^TA_i) = 2smI_m, \tag{3}$$

$$JA_i = A_iJ = aJ, \quad i = \{1, \cdots, s\}, \ a \text{ constant}. \tag{4}$$

Call $A_1, \cdots, A_s$ a regular s-set of matrices of order m if (1), (2), (3) are satisfied and a regular s-set of regular matrices if (4) is also satisfied, these matrices were first discovered by J. Seberry and A. L. Whiteman in "New Hadamard matrices and conference matrices obtained via Mathon's construction", *Graphs and Combinatorics*, 4(1988), 355-377. In this paper, we prove that
(i) if there exist a regular s-set of order m and a regular t-set of order n then there exists a regular s-set of order mn when $t = sm$,
(ii) if there exist a regular s-set of order m and a regular t-set of order n then there exists a regular s-set of order mn when $2t = sm$ (m is odd),
(iii) if there exist a regular s-set of order m and a regular t-set of order n then there exists a regular $2s$-set of order mn when $t = 2sm$.
As applications, we prove that if there exist a regular s-set of order m there exists
(iv) an Hadamard matrix of order $4hm$ whenever there exists an Hadamard matrix of order $4h$ and $s = 2h$,
(v) Williamson type matrices of order nm whenever there exists Williamson type matrices of order n and $s = 2n$,
(vi) a complex Hadamard matrix of order $2cm$ whenever there exists a complex Hadamard matrix of order $2c$ and $s = 2c$.
This paper extends and improves results of Seberry and Whiteman giving new classes of Hadamard matrices, Williamson type matrices and complex Hadamard matrices. In particular, we show that if both $p \equiv 1 \pmod 4$ and $2p + 1$ are prime powers then there exist Williamson type matrices of order $\frac{1}{2}(p + 1)(2p + 1)^2$.

1 Introduction and Basic Definitions

This paper uses sets of matrices first introduced by Seberry and Whiteman [8] to find new classes of Hadamard matrices, Williamson type matrices, orthogonal designs and complex Hadamard matrices.

Definition 1 Suppose $A_1, \cdots, A_s$ are $(1,-1)$ matrices of order m satisfying

$$A_i A_j = J, \quad i,j \in \{1, \cdots, s\}, \tag{5}$$

$$A_i^T A_j = A_j^T A_i = J, \quad i \neq j, \quad i,j \in \{1, \cdots, s\}, \tag{6}$$

$$\sum_{i=1}^{s} (A_i A_i^T + A_i^T A_i) = 2smI_m, \tag{7}$$

$$JA_i = A_i J = aJ, \quad i = \{1, \cdots, s\}, \ a \text{ constant}. \tag{8}$$

Call $A_1, \cdots, A_s$ a *regular s-set of matrices* of order m if (5), (6), (7) are satisfied (see [**?**], [8]) and a *regular s-set of regular matrices* if (8) is also satisfied.

J. Seberry and A. L. Whiteman [8] proved that if if $q \equiv 3 \pmod 4$ is a prime power there exists a regular $\frac{1}{2}(q+1)$-set of regular matrices of order q^2, say A_i, $i = 1, \cdots, \frac{1}{2}(q+1)$ satisfying $A_i J = JA_i = qJ$.

Definition 2 Four $(1,-1)$ matrices X_1, X_2, X_3, X_4 of order n satisfying

$$X_1 X_1^T + X_2 X_2^T + X_3 X_3^T + X_4 X_4^T = 4nI_n$$

and

$$UV^T = VU^T,$$

where $U, V \in \{X_1, X_2, X_3, X_4\}$ will be called *Williamson type matrices.*

Williamson and Williamson type matrices are discussed extensively by Baumert, Hall, Sawade, Miyamoto, Seberry, Whiteman, Yamada and Yamamoto ([**?**], [1], [4], [5], [**?**], [15], [16], [7], [10], [6], [12], [13], [3], [14], [9]).

Definition 3 A matrix $M = (m_{ij})$ of order m satisfying $m_{i,j} = m_{1,j-i+1}$, where the subscripts are reduced modulo m, is called a *circulant matrix.* If $m_{i,j} = m_{1,i+j-1}$, M is called a *back-circulant matrix.*

Definition 4 An *orthogonal design* A, of order p and type $(s_1, \cdots, s_u)$, denoted by $OD(p; s_1, \cdots, s_u)$, on the commuting variables $\pm x_1, \cdots, \pm x_u, 0$ is a matrix of order p with entries $\pm x_1, \cdots, \pm x_u, 0$ satisfying

$$AA^T = (s_1 x_1^2 + \cdots + s_u x_u^2) I_p.$$

Definition 5 Let C be a $(1,-1,i,-i)$ matrix of order c satisfying $CC^* = cI_c$, where C^* is the Hermitian conjugate of C. We call C a *complex Hadamard matrix* of order c.

From [11], any complex Hadamard matrix has order 1 or order divisible by 2. Let $C = X + iY$, where X, Y consist of $1, -1, 0$ and $X \wedge Y = 0$ where $\wedge$ is the Hadamard product. Clearly, if C is a complex Hadamard matrix then $XX^T + YY^T = cI_c$, $XY^T = YX^T$.

2 Product of Two Sets of Matrices

Theorem 1 *If there exist a regular s-set of matrices of order m and a regular $t(= sm)$-set of matrices of order n then there exists a regular s-set of matrices of order mn.*

Proof. Let $\{ A_1 = (a^1_{ij}),\ A_2 = (a^2_{ij}), \cdots,\ A_s = (a^s_{ij}) \}$ be the regular s-set of matrices of order m and $\{ B_1,\ B_2, \cdots,\ B_t \}$ be the regular t-set of matrices of order of n.
Define $C_i = (c^i_{kj}) = (a^i_{kj} B_{(i-1)m+j+k-1})$, $i = 1, \cdots, s$ so that

$$C_i = \begin{bmatrix} a^i_{11}B_{(i-1)m+1} & a^i_{12}B_{(i-1)m+2} & \cdots & a^i_{1m}B_{im} \\ a^i_{21}B_{(i-1)m+2} & a^i_{22}B_{(i-1)m+3} & \cdots & a^i_{2m}B_{(i-1)m+1} \\ & & \vdots & \\ a^i_{m1}B_{im} & a^i_{m2}B_{(i-1)m+1} & \cdots & a^i_{mm}B_{im-1} \end{bmatrix}.$$

Since both $\{A_1, A_2, \cdots, A_s\}$ and $\{B_1, B_2, \cdots, B_t\}$ are regular r-sets of matrices, $r = s, t$ respectively, we have

$$C_iC_j = J_m \times J_n = J_{mn},\ i,j \in \{1, \cdots, s\},$$
$$C_iC_j^T = C_j^T C_i = J_m \times J_n = J_{mn}, i \neq j, i,j \in \{1, \cdots, s\}.$$

To show

$$\sum_{i=1}^{s}(C_iC_i^T + C_i^T C_i) = 2smnI_{mn}, \tag{9}$$

note that $(a^i_{kj})^2 = 1$ so the diagonal element of $C_iC_i^T + C_i^T C_i$ is

$$\sum_{j=1}^{m}(B_{(i-1)m+j}B^T_{(i-1)m+j} + B^T_{(i-1)m+j}B_{(i-1)m+j})$$

and hence the diagonal element of $\sum_{i=1}^{s}(C_iC_i^T + C_i^T C_i)$ is

$$\sum_{j=1}^{sm}(B_jB_j^T + B_j^T B_j) = 2tnI_n = 2smnI_n.$$

The off-diagonal elements of $C_iC_i^T$ are given by

$$\sum_{j=1}^{m}(a^i_{hj}a^i_{kj}B_{(i-1)m+j+h-1}B^T_{(i-1)m+j+k-1}), h \neq k$$

$$= \sum_{j=1}^{m} a^i_{hj}a^i_{kj}J.$$

So the off-diagonal element of $\sum_{i=1}^{s}(C_iC_i^T + C_i^T C_i)$, taking into account diagonal elements of (9) for $A_1, \cdots, A_s$, is zero,

$$\sum_{i=1}^{s}\sum_{j=1}^{m}(a^i_{hj}a^i_{kj} + a^i_{jh}a^i_{jk})J = 0.$$

□

We also note that if $B_jJ_n = bJ_n$ and $A_iJ_m = aJ_m$, (part(8) of the Definition 1), then

$$(\sum_{j=1}^{m} a_{kj}^i B_l)J_n = (\sum_{j=1}^{m} a_{kj}^i)bJ_n = abJ_n$$

and $C_iJ_{mn} = abJ_{mn}$. Similarly $J_{mn}C_i = abJ_{mn}$. Thus we have

Corollary 1 *If there exist a regular s-set of regular matrices of order m and a regular $t(= sm)$-set of regular matrices of order n then there exists a regular s-set of regular matrices of order mn.*

We now use a result of Seberry and Whiteman [8] who showed that if $q \equiv 3 \pmod 4$ is a prime power there exists a regular $\frac{1}{2}(q+1)$-set of regular matrices of order q^2.

Corollary 2 *If both $q \equiv 3 \pmod 4$ and $(q+1)q^2-1$ are prime powers then there exists a regular $\frac{1}{2}(q+1)$-set of regular matrices of order $q^2((q+1)q^2-1)^2$.*

Proof. Note $(q+1)q^2 - 1 \equiv 3 (\mathrm{mod}\ 4)$. By [8], there exist both a regular $\frac{1}{2}(q+1)$-set of regular matrices of order q^2 and a regular $\frac{1}{2}(q+1)q^2$-set of regular matrices of order $((q+1)q^2-1)^2$. Using Theorem 1, we have a regular $\frac{1}{2}(q+1)$-set of matrices of order $q^2((q+1)q^2-1)^2$. □

A result of Seberry and Whiteman (see Theorem 12 of [8]) would now give the next Corollary which is new. We shall give another proof of their results in Section 3.

Corollary 3 *If both $q \equiv 3 \pmod 4$ and $(q+1)q^2-1$ are prime powers then there exists an Hadamard matrix of order $q^2(q+1)((q+1)q^2-1)^2$.*

Proof. By Theorem 1, there exists a regular $\frac{1}{2}(q+1)$-set of matrices of order $q^2(q^2(q+1)-1)^2$. On the other hand, from the Index, [**?**], there exists an Hadamard matrix of order $q+1$. Finally, by Theorem 12, [8], we have an Hadamard matrix of order $q^2(q+1)(q^2(q+1)-1)^2$. □

Theorem 2 *If there exist a regular s-set of matrices of order m and a regular t-set of matrices of order n then there exists a regular s-set of matrices of order mn, when $2t = sm$ (m is odd).*

Proof. Let { $A_1 = (a_{ij}^1)$, $A_2 = (a_{ij}^2)$, $\cdots$, $A_s = (a_{ij}^s)$ } be the regular s-set of matrices of order m and { B_1, B_2, $\cdots$, B_t } be the regular t-set of matrices of order n. Note $t = \frac{1}{2}sm$, $\frac{1}{2}s$ is an integer as $2t = sm$ and m is odd.
Set $\frac{1}{2}s = r$. For $i = 1, \cdots, r$, define $C_i = (c_{kj}^i) = (a_{kj}^i B_{(i-1)m+j+k-1})$, note C_i is a matrix of blocks i.e.

$$C_i = \begin{bmatrix} a_{11}^i B_{(i-1)m+1} & a_{12}^i B_{(i-1)m+2} & \cdots & a_{1m}^i B_{im} \\ a_{21}^i B_{(i-1)m+2} & a_{22}^i B_{(i-1)m+3} & \cdots & a_{2m}^i B_{(i-1)m+1} \\ & & \vdots & \\ a_{m1}^i B_{im} & a_{m2}^i B_{(i-1)m+1} & \cdots & a_{mm}^i B_{im-1} \end{bmatrix}$$

and for $i = r+1, \cdots, 2r = s$, $C_i = (c^i_{kj}) = (a^i_{kj} B^T_{(i-1)m+j+k-1})$, i.e.

$$C_i = \begin{bmatrix} a^i_{11} B^T_{(i-1)m+1} & a^i_{12} B^T_{(i-1)m+2} & \cdots & a^i_{1m} B^T_{im} \\ a^i_{21} B^T_{(i-1)m+2} & a^i_{22} B^T_{(i-1)m+3} & \cdots & a^i_{2m} B^T_{(i-1)m+1} \\ & & \vdots & \\ a^i_{m1} B^T_{im} & a^i_{m2} B^T_{(i-1)m+1} & \cdots & a^i_{mm} B^T_{im-1} \end{bmatrix}.$$

Since both $\{A_1, A_2, \cdots, A_s\}$ and $\{B_1, B_2, \cdots, B_t\}$ are regular l-set of matrices, $l = s, t$ respectively, we have

$$C_i C_j = J_m \times J_n = J_{mn}, i, j \in \{1, \cdots, s\},$$

$$C_i C_j^T = C_j^T C_i = J_m \times J_n = J_{mn}, i \neq j, i, j \in \{1, \cdots, s\}.$$

We now prove $\sum_{i=1}^s (C_i C_i^T + C_i^T C_i) = 2smnI_{mn}$. Note that $(a^i_{kj})^2 = 1$ so the diagonal element of $C_i C_i^T + C_i^T C_i$ is

$$\sum_{j=1}^m (B_{(i-1)m+j} B^T_{(i-1)m+j} + B^T_{(i-1)m+j} B_{(i-1)m+j}),$$

for $i = 1, \cdots, r$ and

$$\sum_{j=1}^m (B^T_{(i-1)m+j} B_{(i-1)m+j} + B_{(i-1)m+j} B^T_{(i-1)m+j}),$$

for $i = r+1, \cdots, s$. So the diagonal element of $\sum_{i=1}^s (C_i C_i^T + C_i^T C_i)$ is

$$2 \sum_{j=1}^{rm} (B_j B_j^T + B_j^T B_j) = 2 \sum_{j=1}^{t} (B_j B_j^T + B_j^T B_j) = 2 \cdot 2tnI_n = 2smnI_n.$$

The off-diagonal elements of $C_i C_i^T$ are given by

$$\sum_{j=1}^m (a^i_{hj} a^i_{kj} B_{(i-1)m+j+h-1} B^T_{(i-1)m+j+k-1}), h \neq k$$

$$= \sum_{j=1}^m a^i_{hj} a^i_{kj} J,$$

for $i = 1, \cdots, r$. By the same reasoning, the off-diagonal elements of $C_i C_i^T$ are also

$$\sum_{j=1}^m a^i_{hj} a^i_{kj} J,$$

$h \neq k$, $i = r+1, \cdots, 2r = s$. Hence the off-diagonal element of $\sum_{i=1}^{2r} (C_i C_i^T + C_i^T C_i)$ is zero, using

$$\sum_{i=1}^m \sum_{j=1}^m (a^i_{hj} a^i_{kj} + a^i_{jh} a^i_{jk}) J = 0.$$

□

By the same reason as in the proof for Corollary 1, we have

Corollary 4 *If there exist a regular s-set of regular matrices of order m and a regular t-set of regular matrices of order n then there exists a regular s-set of regular matrices of order mn, when $2t = sm$ (m is odd).*

Corollary 5 *If both $q \equiv 7 \pmod 8$ and $\frac{1}{2}(q+1)q^2 - 1$ are prime powers then there exists a regular $\frac{1}{2}(q+1)$-set of regular matrices of order $q^2(\frac{1}{2}(q+1)q^2-1)^2$.*

Proof. Note $\frac{1}{2}(q+1)q^2 - 1 \equiv 3 \pmod 4$. By [8], there exist both a regular $\frac{1}{2}(q+1)$-set of regular matrices of order q^2 and a regular $\frac{1}{4}(q+1)q^2$-set of regular matrices of order $(\frac{1}{2}(q+1)q^2-1)^2$. Using Theorem 2, we have a regular $\frac{1}{2}(q+1)$-set of regular matrices of order $q^2(\frac{1}{2}(q+1)q^2-1)^2$. □

By the same reasoning as in the proof for Corollary 3, we have

Corollary 6 *If both $q \equiv 7 \pmod 8$ and $\frac{1}{2}(q+1)q^2 - 1$ are prime powers then there exists an Hadamard matrix of order of $q^2(q+1)(\frac{1}{2}(q+1)q^2-1)^2$.*

Theorem 3 *If there exist a regular s-set of matrices of order m and a regular t-set of matrices of order n then there exists a regular $2s$-set of matrices of order mn, when $t = 2sm$.*

Proof. Let $\{ A_1 = (a^1_{ij}),\ A_2 = (a^2_{ij}), \cdots,\ A_s = (a^s_{ij}) \}$ be the regular s-set of matrices of order m and $\{ B_1, B_2, \cdots, B_t \}$ be the regular t-set of matrices of order of n.
Define

$$C_i = \begin{bmatrix} a^i_{11}B_{(i-1)m+1} & a^i_{12}B_{(i-1)m+2} & \cdots & a^i_{1m}B_{im} \\ a^i_{21}B_{(i-1)m+2} & a^i_{22}B_{(i-1)m+3} & \cdots & a^i_{2m}B_{(i-1)m+1} \\ & & \vdots & \\ a^i_{m1}B_{im} & a^i_{m2}B_{(i-1)m+1} & \cdots & a^i_{mm}B_{im-1} \end{bmatrix}$$

and

$$C_{s+i} = \begin{bmatrix} a^i_{11}B_{(i-1)m+1} & a^i_{21}B_{(i-1)m+2} & \cdots & a^i_{m1}B_{im} \\ a^i_{12}B_{(i-1)m+2} & a^i_{22}B_{(i-1)m+3} & \cdots & a^i_{m2}B_{(i-1)m+1} \\ & & \vdots & \\ a^i_{1m}B_{im} & a^i_{2m}B_{(i-1)m+1} & \cdots & a^i_{mm}B_{im-1} \end{bmatrix},$$

$i = 1, \cdots, s$. By the same reasoning as in the proofs for Theorem 1 and Theorem 2, we have

$$C_iC_j = J_m \times J_n = J_{mn}, i, j \in \{1, \cdots, s\},$$

$$C_iC_j^T = C_j^TC_i = J_m \times J_n = J_{mn}, i \neq j, i, j \in \{1, \cdots, s\}.$$

and

$$\sum_{i=1}^{s}(C_iC_i^T + C_i^TC_i) = 2smnI_{mn}.$$

Corollary 7 *If there exist a regular s-set of regular matrices of order m and a regular t-set of regular matrices of order n then there exists a regular $2s$-set of regular matrices of order mn, when $t = 2sm$.*

Corollary 8 *If both $q \equiv 3 \pmod 4$ and $2(q+1)q^2-1$ are prime powers there exists a regular $(q+1)$-set of regular matrices of order $q^2(2(q+1)q^2-1)^2$.*

Proof. Note $2(q+1)q^2-1 \equiv 3 \pmod 4$. By [8], there exist both a regular $\frac{1}{2}(q+1)$-set of regular matrices of order q^2 and a regular $(q+1)q^2$-set of regular matrices of order $(2(q+1)q^2-1)^2$. Using Theorem 3, we have a regular $(q+1)$-set of regular matrices of order $q^2(2(q+1)q^2-1)^2$. □

By the same reasoning as in the proof for Corollary 3, we have

Corollary 9 *If both $q \equiv 3 \pmod 4$ and $2(q+1)q^2-1$ are prime powers there exists an Hadamard matrix of order of $2q^2(q+1)(2(q+1)q^2-1)^2$.*

We note that if $t=2$ in Theorems 1, 2, 3 the conditions $t=sm$, $2t=sm$, $t=2sm$ can be removed and a completely different proof obtained.

3 Hadamard Matrices

We give another proof of Seberry and Whiteman's Theorem [8].

Theorem 4 *If there exists an Hadamard matrices of order $4h$ and a regular $s(=2h)$-set of matrices of order m then there exists an Hadamard matrix of order $4hm$.*

Proof. Let $\{ A_1, \cdots, A_s \}$ be the regular s-set of matrices of order m and $H=(h_{ij})$ be the Hadamard matrix of order $4h$. Set $L_1=(h_{ij}A_{j+i-1})$, $L_2=(h_{i,2h+j}A^T_{j+i-1})$, $L_3=(h_{2h+i,j}A_{j+i-1})$, $L_4=(h_{2h+i,2h+j}A^T_{j+i-1})$, where $i,j=1,\cdots,2h$ and all the subscripts $j+i-1$ are reduced modulo $2h$. Set

$$E = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix}.$$

We now prove E is an Hadamard matrix of order $4hm$. Let

$$E = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_{4h} \end{bmatrix},$$

where $E_1, E_2, \cdots E_{4h}$ are of order $m \times 4hm$. It is easy to verify $E_iE_j^T=0$, if $i \neq j$ and $E_iE_i^i = \sum_{k=1}^{2h}(A_kA_k^T + A_k^TA_k) = \sum_{k=1}^{s}(A_kA_k^T + A_k^TA_k) = 2smI_m = 4hmI_m$. Thus $EE^T = 4hmI_{4hm}$. □

4 Williamson Type Matrices

We find new constructions for Williamson matrices not given by Miyamoto [3] or Seberry and Yamada [?], [9].

Theorem 5 *If there exist Williamson type matrices of order n and a regular $s(=2n)$-set of matrices of order m then there exist Williamson type matrices of order nm.*

Proof. Let $A=(a_{ij})$, $B=(b_{ij})$, $C=(c_{ij})$, $D=(d_{ij})$ be the Williamson type matrices of order n and $\{\ R_1,\cdots,R_s\ \}$ be the regular s-set of matrices of order m. Set $E=(a_{ij}R_{j+i-1})$, $F=(b_{ij}R_{n+j+i-1})$, $G=(c_{ij}R^T_{j+i-1})$, $H=(d_{ij}R^T_{n+j+i-1})$, where $i,j=1,\cdots,n$ and the subscripts $j+i-1$ are reduced modulo n. It is easy to show $UV^T=VU^T$, for $U,V\in E,F,G,H$. We now prove

$$EE^T+FF^T+GG^T+HH^T=4mnI_{mn}.$$

Let $E=\begin{bmatrix}E_1\\E_2\\\vdots\\E_n\end{bmatrix}$, Let $F=\begin{bmatrix}F_1\\F_2\\\vdots\\F_n\end{bmatrix}$, Let $G=\begin{bmatrix}G_1\\G_2\\\vdots\\G_n\end{bmatrix}$, Let $H=\begin{bmatrix}H_1\\H_2\\\vdots\\H_n\end{bmatrix}$, where E_i, F_i, G_i, H_i are of order $m\times mn$. By the conditions of Williamson type matrices and (1), (2), (3), it is easy to verify that if $i\neq j$, $E_iE_j^T+F_iF_j^T+G_iG_j^T+H_iH_j^T=0$. On the other hand, $E_iE_i^T+F_iF_i^T+G_iG_i^T+H_iH_i^T=\sum_{k=1}^{2n}(R_kR_k^T+R_k^TR_k)=\sum_{k=1}^{s}(R_kR_k^T+R_k^TR_k)=2smI_m=4mnI_m$. Thus $EE^T+FF^T+GG^T+HH^T=4mnI_{mn}$. $\square$

We give a new proof of construction for Williamson type matrices of [8 , 10]:

Corollary 10 *If n is the order of Williamson type matrices and $4n-1$ is a prime power then there exist Williamson type matrices of order $n(4n-1)^2$.*

Proof. Clearly, $4n-1\equiv 3 \pmod 4$. By [8], there exists a regular $2n$-set of regular matrices of order $(4n-1)^2$. From Theorem 4, we have Williamson type matrices of order $n(4n-1)^2$. $\square$

The next construction appears to be new:

Corollary 11 *If both $p\equiv 1 \pmod 4$ and $2p+1$ are prime powers then there exist Williamson type matrices of order $\frac{1}{2}(p+1)(2p+1)^2$.*

Proof. From the Index of [**?**], there exist Williamson matrices of order $\frac{1}{2}(p+1)$. Using Corollary 10, we have Williamson type matrices of order $\frac{1}{2}(p+1)(2p+1)^2$. $\square$

Let $p=17$, we have new Williamson type matrices of order 11025. Let $p=21,25$ we have new construction for known Williamson type matrices of order 20339, 33813.

Corollary 12 *If $28\cdot 3^i-1$ is a prime power then there exist Williamson type matrices of order $7\cdot(28\cdot 3^i-1)^2 3^i$, where $i,=0,1,\cdots$.*

Proof. From the Index of [**?**], there exist Williamson type matrices of order of $7\cdot 3^i$, where $i=0,1,\cdots$. By Corollary 10, we have Williamson type matrices of order $7\cdot(28\cdot 3^i-1)^2 3^i$. $\square$

5 Complex Hadamard Matrices

Theorem 6 *If there exist a complex Hadamard matrix of order $2c$ and a regular $s(=2c)$-set of matrices of order m then there exists a complex Hadamard matrix of order $2cm$.*

Proof. Let $\{ A_1, \cdots, A_s \}$ be the regular $s(=2c)$-set of matrices of order m and $C = X + iY$ be the complex Hadamard matrix of order $2c$, where both X and Y are $(1,-1)$ matrices satisfying $X \wedge Y = 0$, $XX^T + YY^T = 2cI_{2c}$, $XY^T = YX^T$. Let $P = X + Y$ and $Q = X - Y$. Then both P and Q are $(1,-1)$ matrices of order $2c$ and $PP^T + QQ^T = 4cI_{2c}$, $PQ^T = QP^T$. Let $P = (p_{ij})$ and $Q = (q_{ij})$, $i, j = 1, \cdots, 2c$. Set $E = (p_{ij}A_{i+j-1})$ and $F = (q_{ij}A_{i+j-1})^T$, where $i, j = 1, \cdots, s$ and the subscripts $i + j - 1$ are reduced modulo s. Clearly, both E and F are $(1,-1)$ matrices of order $2cm$, since both P and Q are $(1,-1)$ matrices of order $2c$. We now prove $EE^T + FF^T = 4cmI_{2cm}$. Rewrite $E = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{bmatrix}$ and $F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix}$, where E_i and F_i are matrices of order $m \times sm$. Note

$$E_iE_i^T + F_iF_i^T = \sum_{j=1}^{s}(p_{ij}p_{ij}A_{i+j-1}A_{i+j-1}^T + q_{ij}q_{ij}A_{i+j-1}^TA_{i+j-1})$$

$$= \sum_{j=1}^{s}(A_jA_j^T + A_j^TA_j) = 2smI_m.$$

On the other hand, if $i \neq k$,

$$E_iE_k^T + F_iF_k^T = \sum_{j=1}^{s}(p_{ij}p_{kj}A_{i+j-1}A_{k+j-1}^T + q_{ij}q_{kj}A_{i+j-1}^TA_{k+j-1})$$

$$= \sum_{j=1}^{s}(p_{ij}p_{kj} + q_{ij}q_{kj})J_m = 0.$$

Thus

$$EE^T + FF^T = 2smI_{sm} = 4cmI_{2cm}.$$

Finally, Set $U = \frac{1}{2}(E+F)$ and $V = \frac{1}{2}(E-F)$. Note both E and F are $(1,-1)$ matrices of order $2cm$ then both U and V are $(1,-1,0)$ matrices of order $2cm$ satisfying $U \wedge V = 0$, $UU^T + VV^T = \frac{1}{2}(EE^T + FF^T) = 2cmI_{2cm}$. Since $PQ^T = QP^T$, $EF^T = FE^T$ and $UV^T = VU^T$. Thus $U + iV$ is a complex Hadamard matrix of order $2cm$. □

Corollary 13 *If both $p \equiv 1 \pmod 4$ and $2p^j(p+1) - 1$ are prime powers then there exists a complex Hadamard matrix of order $p^j(p+1)(2p^j(p+1)-1)^2$, where $j = 1, 2, \cdots$.*

Proof. Obviously, $2p^j(p+1) - 1 \equiv 3 \pmod 4$. By [8], there exists a regular $p^j(p+1)$-set of matrices of order $(2p^j(p+1)-1)^2$. From Corollary 18, [2], there exists a complex Hadamard matrix of order $p^j(p+1)$. Using Theorem 6, we have a $p^j(p+1)(2p^j(p+1)-1)^2$.

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