Some Orthogonal Matrices Constructed by Strong Kronecker Multiplication

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Abstract

Strong Kronecker multiplication of two matrices is useful for constructing new orthogonal matrices from known those. In this paper we give strong Kronecker multiplication a general form and a short proof. To show its applications, we prove that if there exists a complex Hadamard matrix of order $2c$ then there exists
(i) a $W(4nc, 2kc)$, if there exists a $W(2n, k)$,
(ii) a complex Hadamard matrix of order $4hc$, if there exists an Hadamard matrix of order $4h$,
(iii) Williamson matrices of order $2cn$, if there exist Williamson matrices of order n,
(iv) an $OD(4cn; 2cs_1, \cdots, 2cs_u)$, if there exists an $OD(2n; s_1, \cdots, s_u)$.
Also we generalize the above results by using more complex orthogonal matrices.

1 Introduction and Basic Definitions

Definition 1 Let C be a $(1, -1, i, -i)$ matrix of order c satisfying $CC^* = cI$, where C^* is the Hermitian conjugate of C. We call C a *complex Hadamard matrix* order c.

From [6], any complex Hadamard matrix has order 1 or order divisible by 2. Let $C = X + iY$, where X, Y consist of $1, -1, 0$ and $X \wedge Y = 0$ where $\wedge$ is the Hadamard product. Clearly, if C is an complex Hadamard matrix then $XX^T + YY^T = cI$, $XY^T = YX^T$.

Definition 2 Let W be a $(1,-1,0)$ matrix of order n satisfying $WW^T = kI_n$. We call W a *weighing matrix* (see [3]) of order n with weight k, denoted by $W = W(n,k)$.

Definition 3 *A complex orthogonal design* (see [2]) , of order n and type $(s_1,\cdots,s_u)$, denoted by $COD(m; s_1, s_2, \cdots, s_u)$ on the commuting variables $x_1,\cdots,x_u$ is a matrix of order n, say X , consists of $e_1x_1,\cdots,e_ux_u,0$, where $e_1,\cdots,e_u \in \{1,-1,i,-i\}$, satisfying

$$XX^* = (\sum_{j=1}^{u} s_j x_j^2) I_n$$

In particular, if $e_1,\cdots,e_u \in \{1,-1\}$, the complex orthogonal will be called an *orthogonal design* denoted by $OD(m; s_1, s_2, \cdots, s_u)$.

Definition 4 Four (1 , -1) matrices A_1, A_2, A_3, A_4 of order n satisfying

$$A_1A_1^T + A_2A_2^T + A_3A_3^T + A_4A_4^T = 4nI_n$$

and

$$UV^T = VU^T,$$

where $U, V \in \{A_1, A_2, A_3, A_4\}$ will be called *Williamson type matrices* of order n (see [**?**]).

Let M be a matrix of order tm. Then M can be expressed as

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ M_{21} & M_{22} & \cdots & M_{2t} \\ & & \vdots & \\ M_{t1} & M_{t2} & \cdots & M_{tt} \end{bmatrix}$$

where M_{ij} is of order m $(i,j = 1,2,\cdots,t)$. Analogously with Seberry and Yamada [5], we call this a t^2 *block M-structure* when M is an orthogonal matrix.

To emphasize the block structure , we use the notation $M_{(t)}$, where $M_{(t)} = M$ but in the form of t^2 blocks, each of which has order m.

Let N be a matrix of order tn. Then, write

$$N_{(t)} = \begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1t} \\ N_{21} & N_{22} & \cdots & N_{2t} \\ & & \cdots & \\ N_{t1} & N_{t2} & \cdots & N_{tt} \end{bmatrix}$$

where N_{ij} is of order n $(i,j = 1,2,\cdots,t)$.

We now define the operation $\bigcirc$ as the following:

$$M_{(t)} \bigcirc N_{(t)} = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1t} \\ L_{21} & L_{22} & \cdots & L_{2t} \\ & & \cdots & \\ L_{t1} & L_{t2} & \cdots & L_{tt} \end{bmatrix}$$

where M_{ij}, N_{ij} and L_{ij} are of order of m, n and mn, respectively and

$$L_{ij} = M_{i1} \times N_{1j} + M_{i2} \times N_{2j} + \cdots + M_{it} \times N_{tj},$$

where $\times$ is Kronecker product, $i, j = 1, 2, \cdots, t$. We call this the *strong Kronecker* multiplication of two matrices.

2 Strong Kronecker Product

In [**?**] the authors prove

Theorem 1 *Let A be an $OD(tm; p_1, \cdots, p_u)$ with entries $x_1, \cdots, x_u$ and B be an $OD(tn; q_1, \cdots, q_s)$ with entries $y_1, \cdots, y_s$. Suppose all $x_1, \cdots, x_u$ and $y_1, \cdots, y_s$ are commutative then*

$$(A_{(t)} \bigcirc B_{(t)})(A_{(t)} \bigcirc B_{(t)})^T = (\sum_{j=1}^{u} p_j x_j^2)(\sum_{j=1}^{s} q_j y_j^2) I_{tmn}.$$

($A_{(t)} \bigcirc B_{(t)}$is not an orthogonal design but an orthogonal matrix.)

We now give Theorem 1 a more general form and a short proof.

Theorem 2 *Let P be a complex $OD(tm; p_1, \cdots, p_u)$ with entries $e_1x_1, \cdots, e_ux_u$ and Q be a complex $OD(tn; q_1, \cdots, q_s)$ with entries $f_1y_1, \cdots, f_sy_s$, where $e_1, \cdots, e_u$, $f_1, \cdots, f_s \in$ { 1, -1, i, -i }. Suppose all $x_1, \cdots, x_u$ and $y_1, \cdots, y_s$ are commutative then*

$$(P_{(t)} \bigcirc Q_{(t)})(P_{(t)} \bigcirc Q_{(t)})^* = (\sum_{j=1}^{u} p_j x_j^2)(\sum_{j=1}^{s} q_j y_j^2) I_{tmn}.$$

($P_{(t)} \bigcirc Q_{(t)}$ is not a complex orthogonal design but a complex orthogonal matrix.)

Proof. Write $P = [P_1 \cdots P_t]$ and $Q = \begin{bmatrix} Q_1 \\ \vdots \\ Q_t \end{bmatrix}$, where $P_1, \cdots, P_t$ are of order $tm \times m$, $Q_1, \cdots, Q_t$ are of order $n \times tn$ From $PP^* = (\sum_{j=1}^{u} p_j x_j^2) I_{tm}$, we have

$$\sum_{j=1}^{t} P_j P_j^* = (\sum_{j=1}^{u} p_j x_j^2) I_{tm}.$$

Since $QQ^* = (\sum_{j=1}^{s} q_j y_j^2) I_{nt}$,

$$Q_i Q_j^* = \begin{cases} (\sum_{j=1}^{s} q_j y_j^2) I_n & \text{if } i = j, \\ 0 & i \neq j. \end{cases}$$

Then

$$\begin{aligned} RR^* &= (\sum_{j=1}^{t} P_j \times Q_j)(\sum_{j=1}^{t} P_j^* \times Q_j^*) \\ &= \sum_{j=1}^{t} (P_j P_j^*) \times (Q_j Q_j^*) \\ &= \sum_{j=1}^{t} (P_j P_j^*) \times (\sum_{j=1}^{s} q_j y_j^2) I_n \\ &= (\sum_{j=1}^{u} p_j x_j^2) I_{mt} \times (\sum_{j=1}^{s} q_j y_j^2) I_n \\ &= (\sum_{j=1}^{u} p_j x_j^2)(\sum_{j=1}^{s} q_j y_j^2) I_{mnt}. \end{aligned}$$

As required. □

Corollary 1 *Let P and Q be the $(\pm 1, \pm i, 0)$ matrices of order tm and tn respectively, satisfying $PP^* = pI_{mt}$ and $QQ^* = qI_{nt}$. Then*

$$(P_{(t)} \bigcirc Q_{(t)})(P_{(t)} \bigcirc Q_{(t)})^* = pqI_{tmn}.$$

Proof. In this case, P is a complex design of order p and type $(x_1 = 1)$ and Q is a complex of order q and type ($y_1 = 1$). □

The strong Kronecker multiplication has potential to yield still more constructionsfor new orthogonal matrices.

3 Weighing Matrices

Theorem 3 *If there exist a W(2n , k) and a complex Hadamard matrix of order 2c there exists a W(4nc , 2kc).*

Proof. Let $W = W(2n,k) = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}$, where W_1, W_2, W_3, W_4 are of order n and $C = X + iY$ be the complex Hadamard matrix of order $2c$, where X, Y are $(1,-1,0)$ matrices of order $2c$ satisfying $X \wedge Y = 0$, $XY^T = YX^T$, $XX^T + YY^T = 2cI_{2c}$. Let $U = \begin{bmatrix} X & Y \\ Y & -X \end{bmatrix}$ and

$$V = \begin{bmatrix} X & Y \\ Y & -X \end{bmatrix} \bigcirc \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}.$$

Then V is a $(1,-1,0)$ matrix of order $4cn$. Since $UU^T = 2cI_{4c}$ and $WW^T = kI_{2n}$, by Theorem 1, $VV^T = 2ckI_{4cn}$. Thus V is a $W(4cn, 2ck)$. □

Theorem 4 *If there exist a W(2n , k) and an Hadamard matrix of order* $4h$ *there exists a W(4nh , 2kh).*

Proof. Let $H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$ be the Hadamard matrix of order $4h$, where H_1, H_2, H_3, H_4 are of order $2h$ and $W = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}$ be the $W(2n, c)$, where W_1, W_2, W_3, W_4 are of order n. Let

$$N = \frac{1}{2}\begin{bmatrix} H_1 + H_2 & H_1 - H_2 \\ H_3 + H_4 & H_3 - H_4 \end{bmatrix}.$$

Then $Z = N_{(2)} \bigcirc W_{(2)}$ is a $(1,-1,0)$ matrix of order $4hn$. Note $NN^T = 2hH_{4h}$ and $WW^T = kI_{2n}$, by Theorem 1, $ZZ^T = 2hkI_{4hn}$. Thus Z is a $W(4nh, 2kh)$. □

4 Complex Hadamard Matrices

Theorem 5 *If there exist an Hadamard matrix of order* $4h$ *and a complex Hadamard matrix of order* $2c$ *there exists a complex Hadamard matrix of order* $4hc$*.*

Proof. Let $H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$ be the Hadamard matrix of order $4h$, where H_1, H_2, H_3, H_4 are of order $2h$ and $C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$ be the complex Hadamard matrix of order $2c$, where C_1, C_2, C_3, C_4 are of order c. Let

$$E = \frac{1}{2}\begin{bmatrix} H_1 + H_2 & H_1 - H_2 \\ H_3 + H_4 & H_3 - H_4 \end{bmatrix} \bigcirc \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}.$$

Then E is a $(1,-1,i,-i)$ matrix of order $4hc$. By Theorem 2, $EE^* = 4hcI_{4hc}$. □

In Theorem 5, if C is a real Hadamard matrix, we have the following result first found by Agayan [1]:

Corollary 2 *If there exist Hadamard matrices of order $4u$ and $4v$ there exists an Hadamard matrix of order $8uv$.*

Theorem 5 gives a series of new complex Hadamard matrices. For example, there exist Hadamard matrices of order $4s$, where $s \in S = \{17, 19, 23, 29, 31, 41, 43, 53, 61, 73\}$. On the other hand, there exists a complex Hadamard matrix of order $1024 = 2 \cdot 523$, for which no symmetric conference matrix can exist (p.469, [6]). Then by Theorem 5, we have the new complex Hadamard matrices of order $4 \cdot 523 \cdot s$, where $s \in S$. If let $h = 2$ in Theorem 6.1, [6] we also find new Hadamard matrices of order of $8 \cdot 523 \cdot s$, where $s \in S$.

5 Williamson Type Matrices

Theorem 6 *If there exist Williamson type matrices of order n and complex Hadamard matrix of order $2c$ there exist Williamson type matrices of order $2cn$.*

Proof. Let $C = X + iY$ be the complex Hadamard matrix of order $2c$, where X, Y are $(1,-1,0)$ matrices of order $2c$ satisfying $X \wedge Y = 0$, $XY^T = YX^T$, $XX^T + YY^T = 2cI_{2c}$. Let A_1, A_2, A_3, A_4 be the Williamson type matrices of order n. We now give the theorem a direct proof without using Theorem 1 or Theorem 2. Define

$$B_1 = A_1 \times X + A_2 \times Y, B_2 = A_1 \times Y - A_2 \times X, B_3 = A_3 \times X + A_4 \times Y, B_4 = A_3 \times Y - A_4 \times X.$$

Then B_1, B_2, B_3, B_4 are $(1,-1)$ matrices of order $2cn$. It is easy to verify

$$B_1B_1^T + B_2B_2^T + B_3B_3^T + B_4B_4^T = 4nI_n \times 2cI_{2c} = 8ncI_{2nc}$$

and

$$UV^T = VU^T,$$

where $U, V \in \{B_1, B_2, B_3, B_4\}$. Thus B_1, B_2, B_3, B_4 are Williamson type matrices of order $2nc$. □

Theorem 6 gives a series of new Williamson type matrices. For example, there exist Williamson type matrices of order s, where $s \in S = \{17, 19, 23, 29, 31, 41, 43, 53, 61, 73\}$. On the other hand, there exists a complex Hadamard matrix of order $1024 = 2 \cdot 523$, for which no symmetric conference matrix can exist [7, p469]. Then by Theorem 6, we have the new Williamson type matrices of order $2 \cdot 523 \cdot s$, where $s \in S$.

6 Orthogonal Designs

Theorem 7 *If there exists an* $OD(2n; s_1, \cdots, s_u)$ *and a complex Hadamard matrix of order* $2c$ *there exists an* $OD(4cn; 2cs_1, \cdots, 2cs_u)$.

Proof. Let $C = X + iY$ be the complex Hadamard matrix of order $2c$, where X, Y are $(1, -1, 0)$ matrices of order $2c$ satisfying $X \wedge Y = 0$, $XY^T = YX^T$, $XX^T + YY^T = 2cI_{2c}$. Let $D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$ be the $OD(2n; s_1, \cdots, s_u)$ with elements $x_1, \cdots, x_u, 0$, where D_1, D_2, D_3, D_4 are of order n. Let

$$E = \begin{bmatrix} X & Y \\ Y & -X \end{bmatrix} \bigcirc \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}.$$

Then E is of order $4cn$ and consits of $x_1, \cdots, x_u, 0$. Since

$$\begin{bmatrix} X & Y \\ Y & -X \end{bmatrix} \begin{bmatrix} X & Y \\ Y & -X \end{bmatrix}^T = 2cI_{4c}$$

and

$$DD^T = (\sum_{j=1}^{u} s_j x_j^2) I_{2n}.$$

By Theorem 1, we have

$$EE^T = (\sum_{j=1}^{u} 2cs_j x_j^2) I_{4cn}.$$

Thus E is an $OD(4cn; 2cs_1, \cdots, 2cs_u)$. □

Let

$$F = \begin{bmatrix} X & Y \\ Y & -X \end{bmatrix} \bigcirc \begin{bmatrix} D_1 & D_2 \\ -D_3 & -D_4 \end{bmatrix},$$

where X, Y, D_1, D_2, D_3, D_4 are defined as in the proof for Theorem 7. By the same reason, F is also an $OD(4cn; 2cs_1, \cdots, 2cs_u)$. Let $P = \frac{1}{2}(E + F)$ and $Q = \frac{1}{2}(E - F)$. Then

$$P = \begin{bmatrix} X & Y \\ Y & -X \end{bmatrix} \bigcirc \begin{bmatrix} D_1 & D_2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} X \times D_1 & X \times D_2 \\ Y \times D_3 & Y \times D_4 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} X & Y \\ Y & -X \end{bmatrix} \bigcirc \begin{bmatrix} 0 & 0 \\ D_3 & D_4 \end{bmatrix}$$

$$= \begin{bmatrix} Y \times D_1 & Y \times D_2 \\ -X \times D_3 & -X \times D_4 \end{bmatrix}.$$

We note

$$PP^T = \begin{bmatrix} XX^T \times (D_1D_1^T + D_2D_2^T) & XY^T \times (D_1D_1^T + D_2D_2^T) \\ YX^T \times (D_1D_1^T + D_2D_2^T) & YY^T \times (D_1D_1^T + D_2D_2^T) \end{bmatrix}$$

$$= \begin{bmatrix} XX^T & XY^T \\ YX^T & YY^T \end{bmatrix} \times (\sum_{j=1}^{u} s_j x_j^2) I_n.$$

Similarly

$$QQ^T = \begin{bmatrix} YY^T \times (D_3D_3^T + D_4D_4^T) & -YX^T \times (D_3D_3^T + D_4D_4^T) \\ -XY^T \times (D_3D_3^T + D_4D_4^T) & XX^T \times (D_3D_3^T + D_4D_4^T) \end{bmatrix}$$

$$= \begin{bmatrix} YY^T & -YX^T \\ -XY^T & XX^T \end{bmatrix} \times (\sum_{j=1}^{u} s_j x_j^2) I_n.$$

Then

$$PP^T + QQ^T = \begin{bmatrix} XX^T + YY^T & 0 \\ 0 & XX^T = YY^T \end{bmatrix} \times (\sum_{j=1}^{u} s_j x_j^2) I_n$$

$$= 2cI_{4c} \times (\sum_{j=1}^{u} s_j x_j^2) I_n = 2c \times (\sum_{j=1}^{u} s_j x_j^2) I_{4cn}.$$

On the other hand, it is to check $PQ^T = QP^T = 0$. Finally, note $EF^T = (P+Q)(P-Q)^T = PP^T - QQ^T = (P-Q)(P+Q)^T = FE^T$. Thus we have the following result:

Theorem 8 *If there exists an $OD(2n; s_1, \cdots, s_u)$ and a complex Hadamard matrix of order $2c$ there exist*
(i) matrices P and Q of order 4nc with elements $x_1, \cdots, x_u, 0$, satesfying

$$PP^T + QQ^T = 2c \times (\sum_{j=1}^{u} s_j x_j^2) I_{4cn}$$

and $PQ^T = QP^T = 0$,
(ii) two $OD(4cn; 2cs_1, \cdots, 2cs_u)$, say E and F, satisfying $EF^T = FE^T$.

Corollary 3 *If there exist Hadamard matrices of order $4h_1$ and $4h_2$ there exists an $OD(8h_1h_2; 4h_1h_2s_1, \cdots, 4h_1h_2s_u)$, when an $OD(2n; s_1, \cdots, s_u)$ exists.*

Proof. By Theorem 3, [**?**], there exists a complex Hadamard matrix of order $4h_1h_2$. By Theorem 7, we have an $OD(8h_1h_2; 4h_1h_2s_1, \cdots, 4h_1h_2s_u)$. □

Theorem 9 *If there exists an $OD(2n; s_1, \cdots, s_u)$ and a complex Hadamard matrix of order $2c$ there exists an $OD(4cn; 2cs_1, \cdots, 2cs_u)$.*

Proof. Let $H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$ be the Hadamard matrix of order $4h$, where H_1, H_2, H_3, H_4 are of order $2h$ and $D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$ be the $OD(2n; s_1, \cdots, s_u)$ with elements $x_1, \cdots, x_u, 0$, where D_1, D_2, D_3, D_4 are of order n. Let

$$F = \frac{1}{2} \begin{bmatrix} H_1 + H_2 & H_1 - H_2 \\ H_3 + H_4 & H_3 - H_4 \end{bmatrix} \bigcirc \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}.$$

Then F is of order $4hn$ and consists of $x_1, \cdots, x_u$. By Theorem 1, we have

$$FF^T = (\sum_{j=1}^{l} 2hs_j x_j^2) I_{4hn}.$$

Thus F is an $OD(4hn; 2hs_1, \cdots, 2hs_u)$. □

7 Specialized Results

Complex Hadamard matrices are often used in this paper. We ellustrate the power of our results by noting some classes of complex Hadamard matrices.

Lemma 1 *If there exists a conference matrix of order n then there is symmetric Hadamard matrix of order $2n$ and a skew complex Hadamard matrix of order n. Symmetric coference matrices are knowen for the following orders:*

c_1	$p^r + 1$	$p^r \equiv 1 \pmod 4$ *is a prime power*	
c_2	$(h-1)^2 + 1$	h *is the order of a skew Hadamard matrix*	
c_3	$q^2(q-2) + 1$	$q \equiv 3 \pmod 4$ *is a prime power* $q - 2$ *is a prime power*	*rll*
c_4	$5 \cdot 9^{2t+1} + 1$	$t \geq 0$	
c_5	$(n-1)^s + 1$	n *is the order of a conference matrix* $s \geq 2$	

Note: a conference matrix of order n exists only if $n-1$ is the sum of two squares. Skew Hadamard matrices for the following orders:

SI	$2^t \Pi k_i$	t, r_i, all non-negative positive integers $k_i - 1 \equiv 3 \pmod 4$ a prime power.
SII	$(p-1)^u + 1$	p the order of a skew-Hadamard matrix, $u > 0$ an odd integer.
SIII	$2(q+1)$	$q \equiv 5 \pmod 8$ a prime power.
SIV	$2(q+1)$	$q = p^t$ is a prime power with $p \equiv 5 \pmod 8$ and $t \equiv 2 \pmod 4$.
SV	$4m$	$m \in \{$odd integers between 3 and 31 inclusive$\}$
SVI	$mn(n-1)$	n the order of amicable orthogonal designs of types $((1, n-1); (n))$ and nm the order of an orthogonal design of type $(1, m, mn-m-1)$.
SVII	$4(q+1)$	$q \equiv 9 \pmod{16}$ a prime power.
SVIII	$(\mid t \mid +1)(q+1)$	$q = s^2 + 4t^2 \equiv 5 \pmod 8$ a prime power and $\mid t \mid +1$ the order of a skew- Hadamard matrix
SIX	$4(q^2+q+1)$	q a prime power and $q^2+q+1 \equiv 3, 5$ or $7 \pmod 8$ a prime power or $2(q^2+q+1)+1$ a prime power
SX	$2^t q$	$q = s^2 + 4r^2 \equiv 5 \pmod 8$ a prime power and an orthogonal design $OD(2^t; 1, a, b, c, c+ \mid r \mid)$ exists where $1 + a + b + 2c + \mid r \mid = 2^t$ and $a(q+1) + b(q-4) = 2^t$.
SXI	hm	h the order of a skew-Hadamard matrix, m the order of amicable Hadamard matrices.

By Lemma 1, these conference matrices and skew Hadamard matrices yield complex Hadamard matrices that give the following corollary of Theorem 3, Theorem 5, Theorem 6 and Theorem 7 :

Corollary 4 *Suppose 2c is the order of a symmetric conference matrix. Then there exist*
(i) a $W(4nc, 2uc)$*, whenever* $W(2n, u)$ *exist,*
(ii) complex Hadamard matrices of order $4hc$*, whenever Hadamard matrices of order* $4h$ *exist,*

(iii) Williamson type matrices of order $2nc$, whenever Williamson type matrices of order n exist,
(iv) an $OD(4cn; 2cs_1, \cdots, 2cs_u)$, whenever $OD(2n; s_1, \cdots, s_u)$ exist.

Proof. Use Theorem 3, 5, 6, 7 and Lemma 1. □

Kharagani and Seberry [4] have found complex Hadamard matrices in many other cases. For example, from Corollary 18, [4] there exists a complex Hadamard matrix of order $p^j(p+1)$, when $p \equiv 1 \pmod 4$ or $p+1$ is the order of a symmetric conference matrices. Seberry also found complex Hadamard matrices of order $w(w-1)$ whenever there is a skew complex Hadamard matrix of order w (see [?]).

8 Remark

Actually most of the above constructions rely on two $(1, -1, 0)$ matrices, say X and Y of order n satisfying $X \wedge Y = 0$, $XY^T = YX^T$, $XX^T + YY^T = kI_n$. In this case, $X + iY$ can be called a *complex weighing matrix* (see [2]) of order n and weight k, denoted by $CW(n, k)$.

Theorem 10 *Suppose there exists a CW(2c , r), then there exists*
(i) a W(4nc , rk) if W(2n , k) exists,
(ii) a CW(4hc , 2hc) if an Hadamard matrix of order $4h$ exists,
(iii) an $OD(4cn; rs_1, \cdots, rs_u)$ if an $OD(2n; s_1, \cdots, s_u)$ exists.
(iv) CW(4nc , kr) if CW(2n , k) exists.

Proof. The proofs for (i), (ii), (iii) are the same as the proofs for Theorem 3, Theorem 5, Theorem 7. As for (iv), by simple verification, we have

$$CW(2n, k) \times CW(2c, r) = CW(4nc, kr).$$

□

Theorem 11 *If there exist a $COD(m; s_1, s_2, \cdots, s_u)$ and a $W(2n, k)$ there exists an $OD(2mn; ks_1, ks_2, \cdots, ks_u)$.*

proof. Let $A = U + iV$ be the $COD(m; s_1, s_2, \cdots, s_u)$ where U, V are matrices of order m with elements $x_1, \cdots, x_u, 0$ satisfying $U \wedge V = 0$, $UV^T = VU^T$, $UU^T + VV^T = (\sum_{j=1}^{u} s_j x_j^2) I_n$. Let $W = W(2n, k) = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}$, where W_1, W_2, W_3, W_4 are of order n. Set

$$B = \begin{bmatrix} U & V \\ V & -U \end{bmatrix} \bigcirc \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}.$$

Then B consists of $x_1, \cdots, x_u, 0$. Note

$$\begin{bmatrix} U & V \\ V & -U \end{bmatrix} \begin{bmatrix} U & V \\ V & -U \end{bmatrix}^T = (\sum_{j=1}^{u} s_j x_j^2) I_{2m}.$$

and by Theorem 2,

$$BB^* = k(\sum_{j=1}^{u} s_j x_j^2) I_{2mn}.$$

Then B is an $OD(2mn; ks_1, ks_2, \cdots, ks_u)$. □

Corollary 5 *If there exists a $COD(m; s_1, s_2, \cdots, s_u)$ and an Hadamard matrix of order $4h$ then there exists an $OD(4hm; 4hs_1, 4hs_2, \cdots, 4hs_u)$.*

Proof. In Theorem 11, let $W(2n, k) = W(4h, 4h)$. □

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