# The Product of Four Hadamard Matrices

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**Abstract**

We prove that if there exist Hadamard matrices of order $4m$, $4n$, $4p$, $4q$ then there exists an Hadamard matrix of order $16mnpq$. This improves and extends the known result of Agayan that there exists an Hadamard matrix of order $8mn$ if there exist Hadamard matrices of order $4m$ and $4n$.

## 1 Introduction and Basic Definitions

A *weighing matrix* [1] of order $n$ with weight $k$, denoted $W = W(n,k)$, is a $(1,-1,0)$ matrix satisfying $WW^T = kI_n$. $W(n,n)$ is an Hadamard matrix.

Let $M$ be a matrix of order $tm$. Then $M$ can be expressed as

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ M_{21} & M_{22} & \cdots & M_{2t} \\ & & \vdots & \\ M_{t1} & M_{t2} & \cdots & M_{tt} \end{bmatrix}$$

where $M_{ij}$ is of order $m$ $(i,j = 1,2,\cdots,t)$. Analogously with Seberry and Yamada [2], we call this a $t^2$ *block M-structure* when $M$ is an orthogonal matrix.

To emphasis the block structures, we use the notation $M_{(t)}$, where $M_{(t)} = M$ but in the form of $t^2$ blocks, each of which has order $m$.

Let $N$ be a matrix of order $tn$. Then, write

$$N_{(t)} = \begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1t} \\ N_{21} & N_{22} & \cdots & N_{2t} \\ & & \cdots & \\ N_{t1} & N_{t2} & \cdots & N_{tt} \end{bmatrix}$$

where $N_{ij}$ is of order $n$ $(i, j = 1, 2, \cdots, t)$.

We now define the operation $\bigcirc$ as the following:

$$M_{(t)} \bigcirc N_{(t)} = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1t} \\ L_{21} & L_{22} & \cdots & L_{2t} \\ & & \cdots & \\ L_{t1} & L_{t2} & \cdots & L_{tt} \end{bmatrix}$$

where $M_{ij}$, $N_{ij}$ and $L_{ij}$ are of order of $m, n$ and $mn$, respectively and

$$L_{ij} = M_{i1} \times N_{1j} + M_{i2} \times N_{2j} + \cdots + M_{it} \times N_{tj},$$

$i, j = 1, 2, \cdots, t$. We call this the *strong Kronecker* multiplication of two matrices.

**Lemma 1** *Let $A$ and $B$ be the matrices of order $tm$ and $tn$ respectively, consist of $1, -1, 0$ satisfying $AA^T = pI_{mt}$ and $BB^T = qI_{nt}$. Then*

$$(A_{(t)} \bigcirc B_{(t)})(A_{(t)} \bigcirc B_{(t)})^T = pqI_{tmn}$$

*Proof.* This is Corollary 1, [**?**].

The following two Lemmas prove the main result. The proof of Lemma 2 in [**?**] uses Lemma 1.

**Lemma 2** *If there exist Hadamard matrices of order $4m$ and $4n$ then there exist two disjoint $W(4mn, 2mn)$, $X$ and $Y$, satisfying*
*(i) $X \wedge Y = 0$,*
*(ii) $X \pm Y$ is a $(1, -1)$ matrix,*
*(iii) $XY^T = XY^T$.*

**Lemma 3** *If there exist Hadamard matrices of order $4p$ and $4q$ then there exist two $(1, -1)$ matrices, $S$ and $R$ of order $4pq$, satisfying*
*(i) $SS^T + RR^T = 8pqI_{4pq}$,*
*(ii) $SR^T = RS^T = 0$.*

[**?**] proves Lemma 2 by using strong Kronecker multiplication. In [**?**], which proves Lemma 3 and the equivalent of Lemma 2 and Lemma 3, $S$ and $R$ of Lemma 3 are called an *orthogonal pair*.

We now reprove Lemma 3 from Lemma 2. By Lemma 2, there exist two $W(4pq, 2pq)$, $X$ and $Y$, satisfying $X \wedge Y = 0$, $X \pm Y$ is a $(1, -1)$ matrix, $XY^T = YX^T$. Let $S = X + Y$, $R = X - Y$. Then both $S$ and $R$ are $(1, -1)$ matrices of order $4pq$. Note

$$SS^T + RR^T = 2(XX^T + YY^T) = 8pqI_{4pq}$$

and

$$SR^T = XX^T - YY^T = 0.$$

Similarly, $RS^T = 0$. So $S$ and $R$ are the required matrices for Lemma 3.

## 2 Main Result

**Theorem 1** *If there exist Hadamard matrices of order $4m$, $4n$, $4p$, $4q$ then there exists an Hadamard matrix of order $16mnpq$.*

*Proof.* By Lemma 2, there exist two $W(4mn, 2mn)$, $X$ and $Y$, satisfying (i), (ii), (iii) in Lemma 2. By Lemma 3, there exist two $(1,-1)$ matrices $S$ and $R$ of order $4pq$ satisfying (i) and (ii) in Lemma 3.

Let $H = X \times S + Y \times R$. Then $H$ is a $(1,-1)$ matrix and

$$HH^T = XX^T \times SS^T + YY^T \times RR^T = 2mnI_{4mn}(SS^T + RR^T)$$

$$= 2mnI_{4mn} \times 8pqI_{4pq} = 16mnpqI_{16mnpq}.$$

Thus $H$ is the required Hadamard matrix.

Theorem 1 gives an improvement and extension for the result of Agayan and [**?**] that if there exist Hadamard matrices of order $4m$ and $4n$ then there exists an Hadamard matrix of order $8mn$. Using the result of Agayan repeatedly on four Hadamard matrices of order $4m$, $4n$, $4p$, $4q$, gives an Hadamard matrix of order $32mnpq$.

## References


[1] Geramita, A. V., and Seberry, J. *Orthogonal Designs: Quadratic Forms and Hadamard Matrices.* Marcel Dekker, New York-Basel, 1979.

[2] Seberry, J., and Yamada, M. On the products of Hadamard matrices, Williamson matrices and other orthogonal matrices using M-structures. *JCMCC 7* (1990), 97–137.